VIBRATIONS

Introduction

Vibrations are oscillations that take place at regular intervals of time. Free vibrations take place, when a system is disturbed from equilibrium position, due to the resistance from internal forces. Typically, an elastic spring or elastic nature of a member in a system offers the resistance.

Energy in free vibrations: To disturb a system, some external energy is supplied to the system. At any time during vibration, this energy is present as combination of Potential Energy (PE) and Kinetic Energy (KE) . Sum of the PE (due to elastic elements) and KE (due to mass or inertia) is equal to this external energy, when there is no dissipation of energy due to damping etc.

Conservative system: A system in which energy is conserved (ie., not lost or dissipated) during a cycle of vibration is called a conservative system.

Classification

(Linear – Non-linear)

A fundamental distinction or classification can be made, in mathematical terms, as linear or non-linear. Dynamics involves inertial elements with mass or MI (representing distibution of mass about an axis of rotation). Forces or toques involve second order derivatives of displacements (linear or angular). Thus equation governing motion of the system (displacement) will be atleast a second order differntial equation.
When system dynamics are modelled as a linear differntial equation, then the system is called a Linear system, and associated vibrations as linear vibrations. Opposed to this is the description by a model which is a non-linear differential equation.

Non-linear terms arise due to non-linear input-ouput relations in the description of behaviour of componets used in the system. However, such behaviour is either simplified as linear, or assumed to be in a small range of operation, so that the overall model is treated /justified as a linear equation.

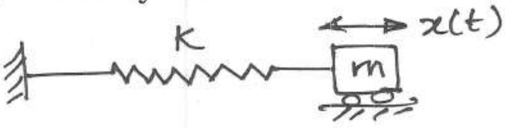
In the accompanying notes, we restrict our study to linear system models and linear vibrations.

Vibrations (linear) **can be classified based on**

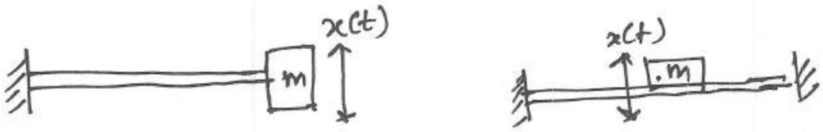
a. Nature of motion *(Longitudinal- Transverse - Torsional)*
b. Whether external force applied *(Free – Forced)*
c. Degrees of freedom *(Single – Two - multi d.o.f)*

a. *Natue of motion:*

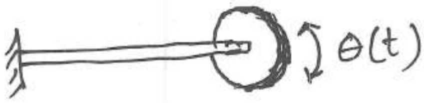
1. Longitudinal : When mass moves along the length / axis of the elastic element. Eg. Spring – mass system



2. Transverse : When mass vibrates perpendicular to the longitudinal axis of elastic element. Eg. Cantilever with mass at end or a simply supported or fixed beam with a mass in the span.



3. Torsional : Rotational oscillation /vibration of a rotor about the axis of the shaft on which rotor / disc is mounted.

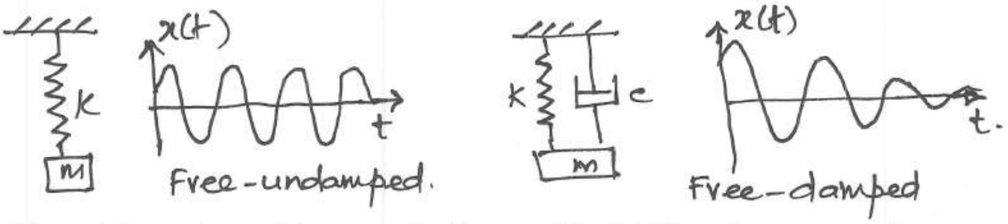


b. *Presence of External force*

Free vibration: (No external force)

-- Free undamped vibration: No damping element is present. Or the real / practical damping is consisdered as small and is ignored. Here the oscillations theoretically go on with the same amplitude.

-- <u>Free damped vibration</u>: When a damping element is present. This reduices the amplitude with each oscillation as energy is lost in each cycle.



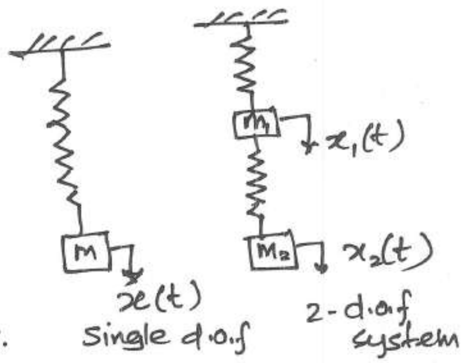
Forced Vibration : An external harmonic force , like **F Sinωt** acts on the system. The system elements vibrate at the same frequency as the external frequency ω.

In this case, the problem of interest is the variation of amplitude with changing external forcing frequency ω, and sometimes the phase difference (ϕ) between the two harmonics, ie., external force and system response.

Our interest is study of variation of system amplitude and ϕ w.r.t the changes in ω, as well as with the damping ratio.

c. *<u>Degrees of freedom</u>*:

The number of independent parameters required to desribe the dynamics or motion of the ements of a system w.r.t time, is equal to the degrees of freedom.



In a spring mass system shown, we can obtain an expression for $x_1(t)$ which describes the motion of the system, ie any point on spring as well. Consider the system with two masses and springs as shown. The system motion now needs the description of two parameters $x_1(t)$ and $x_2(t)$, which are independent of each ther. The number of independent parameters needed to describe are 2. Hence this is called a two degrees of freedom system. Similarly if two rotors are mounted on a shaft, the two parameters $\theta_1(t)$ and $\theta_2(t)$ are independent. Hence it is a 2-d.o.f system.

Similarly 3 d.o.f system etc.

DOF and natural frequencies: A single dof system has one natural frequency like $(k/m)^{1/2}$ for a spring mass system.

A 2 dof system has 2 natural frequencies, ω_{n1} and ω_{n2}. Any arbirary disturbnce leads to periodic motion of x_1 and x_2 each having components of both harmonics. Eg. $X_1(t) = A_1 \,\mathrm{Sin}(\omega_{n1}t + \phi_1) + A_2 \,\mathrm{Sin}(\omega_{n2}t + \phi_2)$. Similarly $x_2(t)$.

In general, the study of free vibrations is mainly to determine the natural frequency and damping ratio. Nat. frequency is indicative of the frequency of oscillations and damping ratio helps to understand how soon the oscillations stop.

In forced vibration cases, knowledge of nat. frequency is required to ensure that external forcing frequency is reasonably away from ω_n to avoid resonance effects. In forced vibrations we also study the forces transmitted to supports, the effect of damping ratio and forcing frequency, etc so that due care is taken in the design to satisfy the requirements w.r.t. vibrating mechanical systems.

Free Vibrations:

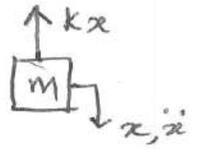
Consider a spring-mass system which is displaced from its equilibrium position

and left. We need to find its governing dynamic equation, the solution of which will be the equation of motion of the mass.

Consider a moment while the mass is oscillating and has dispcament x(t). From the free body diagram (FBD) of the mass, using Newton's equation F = m.a we

get $\quad -k.x = m\,\ddot{x}$

or $\quad m\,\ddot{x} + k\,x = 0.$



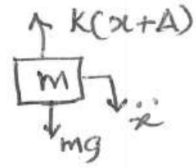
This is a second order differential equation

which is satisfied by ASin ωt or A Cos ωt , when $\omega^2 = k/m$

Here ω is represented as ω_n and is called the natural frequency of the system. It depends upon the values of K and m in the system and is characteristic of the system.

Considering effect of weight

We may write the equation including the displacement of mass due to its weight mg. In this case the spring is stretched by $\Delta = mg / k$



The FBD is as shown, from which

$-k(x + \Delta) + mg = m\ddot{x}$ which becomes, (from $mg = k.\Delta$)

$m\ddot{x} + kx = 0$.

This is the same as when initial deflection is not considered. As it does not affect the governing dynamic equation, we may ignore initial deflection and the corresponding initial potential energy in spring. The equilibrium position in such cases is when it is displaced due to weight. The equation is the same irrespective of whether it is presented in any of the three ways shown below:

Solution of the equation:

For the equation $x + (k/m)x = 0$,

We may write the solution as

$x(t) = A\sin\omega t + B\cos\omega t$ or as

$x(t) = X\,\mathrm{Sin}(\omega t + \phi)$

The two uknowns A and B or (X and ϕ) are found from conditions of x(0) and V(0).

Example: For the spring mass system find expression for displacement x(t), given tht x(0) = 0.2 and v(0) = 0.5.

$\omega = (k/m)^{1/2} = (4000/20)^{1/2} = 10\sqrt{2}$ rad/s

We may take the eqn for displacement of mass as $x(t) = X\,\mathrm{Sin}(\omega t + \phi)$

Which gives, $x(0) = 0.2 = X\,\mathrm{Sin}\,\phi$;

$d/dt\,(x(t)) = \omega X\cos(\omega t + \phi)$ so that

$d/dt(x(0)) = \omega X\,\mathrm{Cos}\,\phi = 0.5$

From above eqns, we get

$\tan\phi = 0.2/0.5\,(10\sqrt{2}) = 5.656$;

$\phi = 80^{o}$ and $X = 0.2/\,\text{Sin}\phi = 0.2/\sin 80^{o} = 0.203$

So that **$x(t) = 0.203\,\text{Sin}(10\sqrt{2}t + 80^{o})$**

Thus the governing dynamic equation of motion obtained from Newton's formulation or other energy methods lead to equation of motion (linear or angular) of the inertial element as shown in equation (2) .

Energy method:

Principle of energy method is that the total energy of the system is constant

(and equal to initial energy given to disturb from equilibrium)

For the spring mass system, when displacement is x,

PE = energy in spring = $kx^2/2$; KE = energy in mass = $mv^2/2$

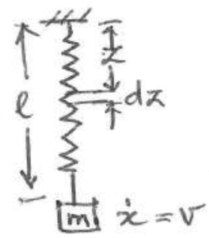
Thus total Energy = T = PE + KE = Constant , so that d/dt(PE+KE) = 0.

In this case we get $d/dt\,(kx^2/2 + m\dot{x}^2/2)$

$$= kx.\dot{x} + m\ddot{x}.\dot{x} = 0$$

Or $m\ddot{x} + kx = 0$, which is the same governing equation.

In the derivation of the governing equation, we may use methods of equilibrium, ie Newton / Euler or Energy method as convenient.



Effect of mass of spring: (m_o)

We had ignored the mass of spring earlier. If the spring mass is significant, we need to consider the same.

At an instant, let the velocity of mass m attached at the end of spring be v. Consider a small element dz at distance z from fixed end, so that KE of the element = $(m_o.\,(dz/l)\,.\,(v.(z/l)^2)/2$

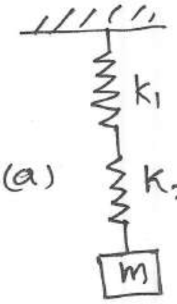
KE of spring = $\int_0^l (m_o.\,(dz/l)\,.\,(v.(z/l)^2/2 = (1/2\,.\,(m_o/3)\,v^2)$

That is, it is like additional mass $m_o/3$ attached to mass m. Hence, the natural frequency, when mass of spring is consisdered is $\omega_n = \sqrt{(k/(m+m_o/3))}$

Natural frequency of systems with spring combinations.

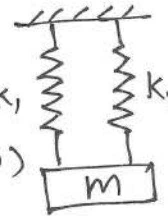
In the following, equivalent spring constant k_e may be substituted in the equation as $\omega_n = \sqrt{\frac{k_e}{m}}$.

(a)



$$K_e = \frac{k_1 k_2}{k_1 + k_2}$$

(b)

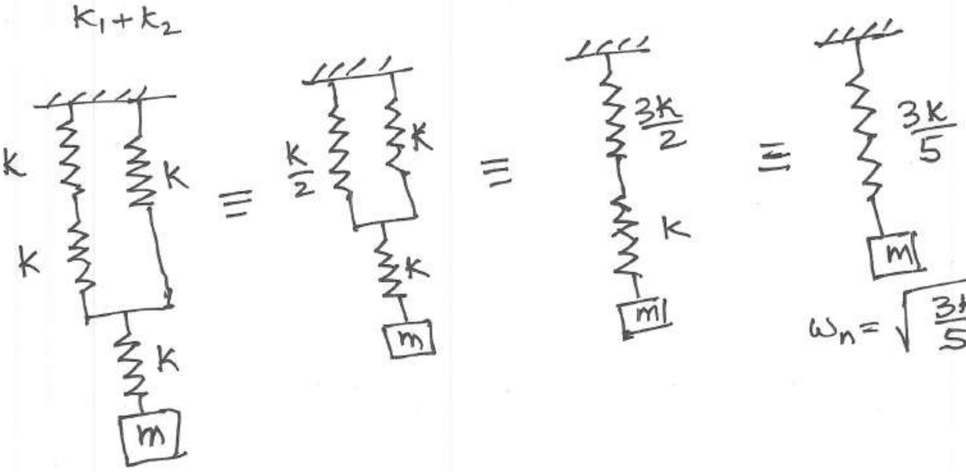


$K_e = k_1 + k_2$

$K_e = k_1 + k_2$

(c)

(d)



$$\omega_n = \sqrt{\frac{3k}{5m}}.$$

Another useful form of ω_n expression.

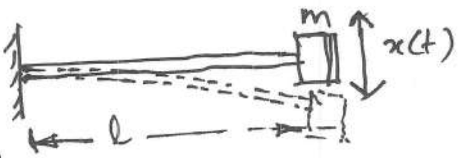
$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{kg}{m.g}} = \sqrt{\frac{g}{mg/k}} = \sqrt{\frac{g}{\Delta}}. \qquad \text{where } \Delta = mg/k.$$

Using the form $\omega_n = \sqrt{\frac{g}{\Delta}}$

Advantage of this form is that ω_n can be estimated from the static deflection observed, and knowledge of k and m is not required.

Transverse Vibrations

- Cantilever with a mass attached at the end as shown has deflection $\Delta = \frac{Wl^3}{3EI}$



Natural frequency of lateral vibration is $\omega_n = \sqrt{K/m}$. as earlier

Here K is the lateral stiffness, or $K = \frac{W}{\Delta} = \frac{3EI}{l^3}$ $(W = mg)$

$\therefore \omega_n = \sqrt{\frac{3EI}{ml^3}}$.

Knowing the material property E, & geometry I and l of beam, ω_n can be estimated for a given 'm'.

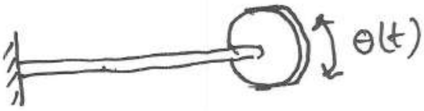
- Note: ω_n can also be calculated from $\omega_n = \sqrt{g/\Delta}$
  where Δ is the deflection observed when m is attached.
  (Again, if Δ is given, E, I, m and l are not required to find ω_n)

- For a simply supported or fixed beam, ω_n can be estimated if deflection Δ is known

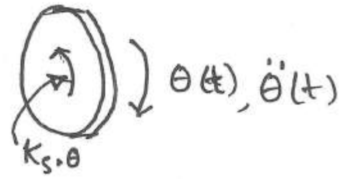
as $\omega_n = \sqrt{g/\Delta}$ [or from $\Delta = \frac{Wl^3}{48EI}$
so that $K = \frac{W}{\Delta} = \frac{48EI}{l^3}$ substituting in $\omega_n = \sqrt{K/m}$)

Torsional Vibration:

Consider a rotor or disc attached to a shaft, and given a twist and left. It will then execute torsional vibration, at natural frequency ω_n, which can be arrived at from the equation of motion of the rotor



Consider the freebody diagram of disc.
Rotation θ of disc is resisted by torsional resistance of shaft.
The torque due to shaft is given by $K_s = T/\theta$



T/θ may be obtained from torsion formula. $\frac{T}{J} = \frac{f_s}{r} = \frac{C\theta}{l}$
as $T/\theta = CJ/l$ which can be evaluated from material and geometric properties of shaft.

Equation of motion:

Assuming, $\ddot{\theta}(t)$ is the angular acceleration, at twist $\theta(t)$,
we have $-K_s \cdot \theta = I \cdot \ddot{\theta}$ where I is the M.I of rotor.

or $I\ddot{\theta} + k_s\theta = 0$ a second order diff. eqn.

$\Rightarrow \omega_n = \sqrt{\frac{K_s}{I}}$.

Energy Method:

Assuming oscillatory motion $\theta(t) = A \sin \omega_n t$

$$KE \text{ of rotor} = \frac{1}{2} I \dot{\theta}^2 = \frac{1}{2} I (A\omega \cos \omega_n t)^2$$

$$KE_{max} = \frac{1}{2} I A^2 \omega_n^2$$

$$PE \text{ (shaft)} = \frac{1}{2} k_s \theta^2 = \frac{1}{2} K_s A^2 \sin^2 \omega t$$

$$PE_{max} = \frac{1}{2} K_s A^2$$

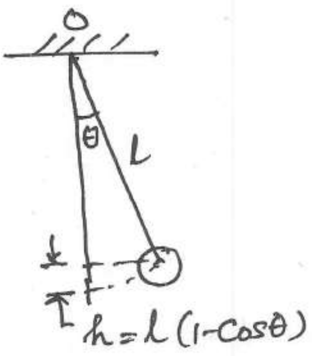
Equating $PE_{max} = KE_{max}$,

$$\frac{1}{2} I A^2 \omega_n^2 = \frac{1}{2} K_s A^2$$

$$\Rightarrow \omega_n = \sqrt{\frac{K_s}{I}}$$

Other Examples

1) Simple Pendulum.



a) Using Equilibrium method:

At angle 'θ' of swing,

Torque due to weight $= I\ddot{\theta}$

or $-mgl\sin\theta = I\ddot{\theta}$

or $I\ddot{\theta} + mgl\theta = 0$ (for small θ)

Here $I = I_G + Ml^2$ (about O).

Neglecting I_G, the MI about its own centre,

$I = Ml^2$

Hence we get $I\ddot{\theta} + Mgl\theta = 0$

$\Rightarrow Ml^2\ddot{\theta} + Mgl\theta = 0$

or $\omega = \sqrt{g/l}$

Using energy method

$T =$ Total Energy $= \frac{1}{2}I\dot{\theta}^2 + mgl(1-\cos\theta)$ (ie KE+PE)

$\frac{d}{dt}(T) = 0 \Rightarrow \omega = \sqrt{g/l}$

Compound Pendulum

The main difference from simple pendulum is $I = I_G + mh^2$

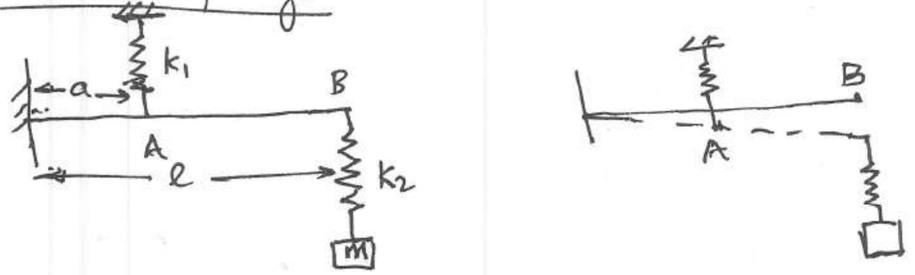
Hence from equilibrium equation

$-mgh\sin\theta = I\ddot{\theta}$

ie $I\ddot{\theta} + mgh\theta = 0$

or $\omega = \sqrt{\frac{mgh}{I}} = \sqrt{\frac{mgh}{I_G + mh^2}}$

Lever with 2 Springs



Deflection of mass at B due to spring k_1

$= (\text{Force at A due to mg})/k_1 = (mg \times \frac{l}{a})/k_1$

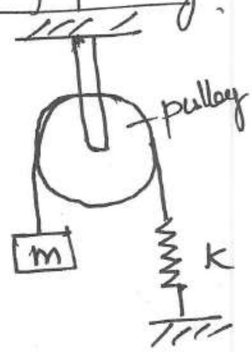
Deflection due to k_1 at $B = \dfrac{(mg\, l/a)}{k_1} \times \dfrac{l}{a} = \dfrac{mg(l/a)^2}{k_1}$

Total deflection at $B = \Delta_1$ due to k_1 + Δ_2 due to k_2

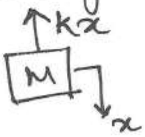
$$\Delta = \frac{mg(l/a)^2}{k_1} + \frac{mg}{k_2}$$

$$\omega = \sqrt{\frac{g}{\Delta}} \Rightarrow \sqrt{\frac{k_1 k_2}{m(k_1 + k_2(l/a)^2)}}$$

Effect of mass of pulley.



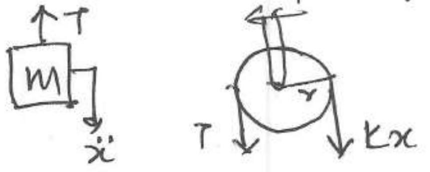
If pulley has no mass, then fbd of mass is



$$\Rightarrow -kx = m\ddot{x}$$

$$\Rightarrow \omega = \sqrt{K/m}$$

If pulley has mass m_0, we need to draw the fbd of mass and pulley (cylindrical)



$x = r\theta$; $\ddot{x} = r\ddot{\theta}$

$I = \dfrac{mr^2}{2}$

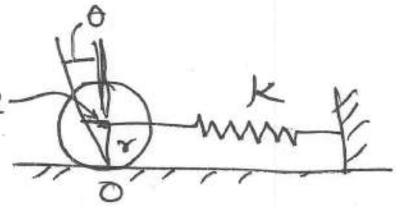
$-T = m\ddot{x}$ & $(T - kx)r = I\ddot{\theta}$

Substituting $T = -m\ddot{x}$, $\quad I\ddot{\theta} + m\ddot{x}r + kxr = 0$

$$\Rightarrow \frac{m_0 r^2}{2}\ddot{\theta} + mr^2\ddot{\theta} + kr^2\theta = 0$$

$$\Rightarrow \omega = \sqrt{\frac{K}{m + m_0/2}}$$

Roller attached to a spring



MI of roller about O

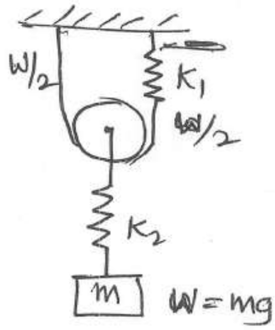
$= \frac{mr^2}{2} + mr^2 = \frac{3}{2} mr^2$

Equilibrium eqn. is $I\ddot{\theta} + kx.r = 0$ $\quad x = r\theta$

or $\frac{3}{2} mr^2 \ddot{\theta} + kr^2\theta = 0$

$\Rightarrow \omega = \sqrt{\frac{2k}{3m}}$

Hanging pulley with springs



Deflection of centre of pulley

$= \frac{1}{2}\left(\frac{\text{Force on spring } k_1}{k_1}\right) = \frac{1}{2} \frac{W/2}{k_1}$

$= W/4k_1$

Total deflection of mass

$= \Delta_{\text{pulley centre}} + \Delta_{\text{due to } k_2}$

$= \frac{W}{4k_1} + \frac{W}{k_2} = mg\left(\frac{k_2 + 4k_1}{4k_1k_2}\right)$

$\Rightarrow \omega = \sqrt{\frac{g}{\Delta}} = \sqrt{\frac{4k_1k_2}{(4k_1+k_2)m}}$

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