

PRINCIPLE OF VIRTUAL WORK

Principle of virtual work is useful in determining the forces or torque required to maintain equilibrium in mechanical systems ie, interconnected rigid bodies.

Principal of virtual work states that virtual work done by external active forces on an ideal mechanical system in equilibrium is zero for any and all virtual displacements consistent with the constraints.

Ideal systems are those composed of two or more rigid members linked together with mechanical connections which are incapable of absorbing energy through compression or elongation and have negligible friction.

In interconnected system three types of forces can be identified.

1. Active forces: External forces capable of doing virtual work during possible virtual displacements.
2. Reactive forces: Forces that act at fixed supports where virtual displacements are not possible. They do no work.
3. Internal forces: Forces in the connection between members. As internal forces act in pairs in opposite directions, the work by one force cancels the work of other force.

Virtual displacement is an assumed hypothetical arbitrary small displacement away from a position of static equilibrium consistent with constraints. Constraint here means restriction of motion imposed by the supports.

The virtual work done on the system by all active forces during a virtual displacement is zero.

If active forces F_1, F_2, \dots in a system go through virtual displacements $\delta x_1, \delta x_2, \dots$ as per constraints, then the total virtual work $\delta U = 0$,

$$\text{Or } \delta U = F_1 \cdot \delta x_1 + F_2 \cdot \delta x_2 + \dots = 0$$

Advantages of the method of virtual work:

1. It is not necessary to dismember the system to obtain relations between the active forces, as in the force and moment summation method.
2. Relations between active forces can be established without reference to reactive (fixed supports) forces.

It is only necessary to obtain the constraining mathematical relation w.r.t. the mechanism.

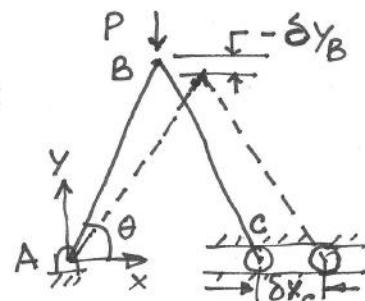
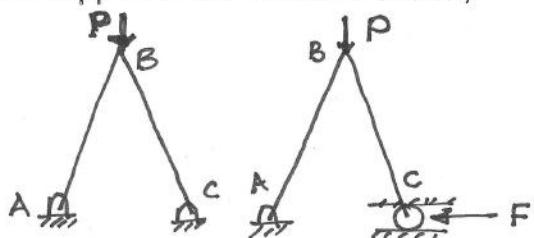
Following examples will illustrate the utility of the method of PVW.

Example 1 (with explanation of convention or signs to be adapted)

Find the horizontal component of the reaction at the support of the structure shown, when a force P is applied.

The structure shown has two hinged supports. To find out the horizontal component at joint B, we assume a roller joint at B free to move along AB only, and assume a horizontal force F acting on B keeping the system in equilibrium. Evaluating F gives the horizontal component of reaction at B.

To apply PVW, we assume a virtual (vertical) displacement at P of δy_B downwards, and the associated displacement of roller at B by δx_C , while maintaining the constraints or geometric relations. The displaced shape of structure after virtual displacements is as shown.



Now the active forces are P and F as they are the only ones that have done the virtual work. Reaction at support joint A does no work and the internal forces at joint B between members does no work.

Virtual work due to P is $\delta U_P = P \cdot \delta y_B$ and that due to F is $\delta U_F = F \cdot \delta x_C$.

The total virtual work $= \delta U = \delta U_P + \delta U_F = 0$.

Sign Convention:

If a force F moving by distance δs , (ie

Work done = $+ F \cdot \delta s$ if F and δs are in the same direction.

= $- F \cdot \delta s$ if F and δs are in opposite directions.

However, δs depends on the coordinate system chosen. The virtual displacement (δx , or δy or $\delta \theta$) has to be assigned +ve or -ve sign as per the sign of the virtual displacement.

For the example, as per this convention, P and distance moved are in the same direction. Hence Virtual Work $P = \delta U_P = P \cdot \delta s = P \cdot (-\delta y_B) = -P \cdot \delta y_B$

F and displacement δx are in opposite directions, Hence,

Virtual Work by $F = -F \cdot \delta s = -F \cdot \delta x_C$

Thus the equation of virtual work becomes

$$\delta U = -P \cdot \delta y_B - F \cdot \delta x_C = 0 \quad (1)$$

Now δy_B and δx_C are substituted from differentials of equations for y_B and x_C .

$$y_B = l \sin \theta ; \quad \delta y_B = l \cos \theta \cdot \delta \theta$$

$$x_C = 2l \cos \theta; \quad \delta x_C = -2l \sin \theta \cdot \delta \theta$$

Substituting in Eqn.(1), we get

$$-P \cdot \delta y_B - F \cdot \delta x_C = -P \cdot l \cos \theta \cdot \delta \theta + F \cdot 2l \sin \theta \cdot \Delta \theta = 0$$

$$\rightarrow F = (P/2) \cot \theta$$

Example 2: A force P acts on the structure shown.

Find the force F required to act on the roller to keep the system in equilibrium.

Solution :-

This is similar to example 1, in which a hinge joint at C is replaced by a roller.

Virtual work equation, assuming similar virtual displacements is

$$-P \cdot \delta y_B - F \cdot \delta x_c = 0$$

$$\Rightarrow F = \frac{P}{2} \cot \theta.$$

Example 2a: Find F when P acts upwards.

In this case both P and F are acting in the direction of virtual displacements.

$$\text{Hence } \delta U = P \cdot \delta s_1 + F \cdot \delta s_2$$

$$= P \cdot \delta y_B + F \cdot (-\delta x_c)$$

$$= P \cdot \delta y_B - F \delta x_c = 0. \quad (1)$$

$$y_B = l \sin \theta; \quad \delta y_B = l \cos \theta \cdot \delta \theta$$

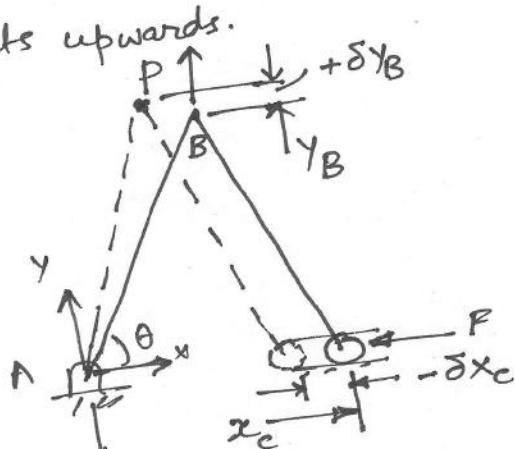
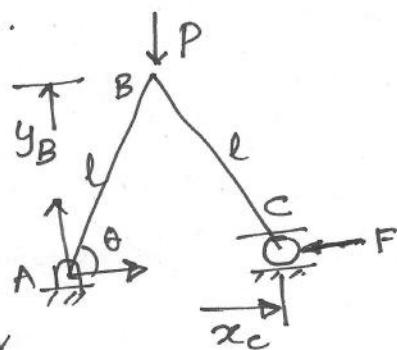
$$x_c = 2l \cos \theta; \quad \delta x_c = -2l \sin \theta \cdot \delta \theta$$

$$\text{Eqn.(1) gives } P l \cos \theta \cdot \delta \theta = F \cdot (-2l \sin \theta \cdot \delta \theta)$$

$$\Rightarrow F = -\frac{P}{2} \cot \theta.$$

F acts in the direction opposite to that assumed.

$F = \frac{P}{2} \cot \theta$ acting to the right. The direction of F obtained is also obvious from the system.

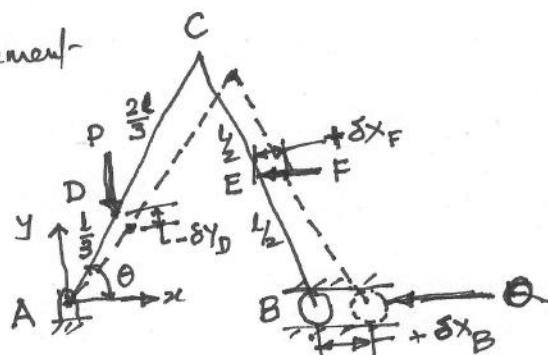


Example: 3

Find the force required to be applied on roller at B, for equilibrium.

Solution:

Assume virtual displacement -
as shown.



The active forces are P, F and Q.

As per sign convention adopted,

$$\delta U = P \cdot \delta s_p - F \cdot \delta s_F - Q \cdot \delta s_Q \\ = -P \delta y_D - F \cdot \delta x_F - Q \cdot \delta x_B = 0 \quad (1)$$

From $y_D = l/3 \sin \theta ; \delta y_D = l/3 \cos \theta \cdot \delta \theta$

$$x_F = \frac{3l}{2} \cos \theta ; \delta x_F = -\frac{3l}{2} \sin \theta \cdot \delta \theta$$

$$x_B = 2l \cos \theta ; \delta x_B = -2l \sin \theta \cdot \delta \theta$$

Substituting in (1), $\rightarrow (2)$

$$P \cdot \frac{l}{3} \cos \theta - F \cdot \frac{3l}{2} \sin \theta - Q \cdot 2l \sin \theta = 0. \quad (2)$$

$$\Rightarrow Q = \frac{P}{6} \cot \theta - \frac{3F}{4}$$

Note: (1) If $F=0$, $Q = \frac{P}{6} \cot \theta$

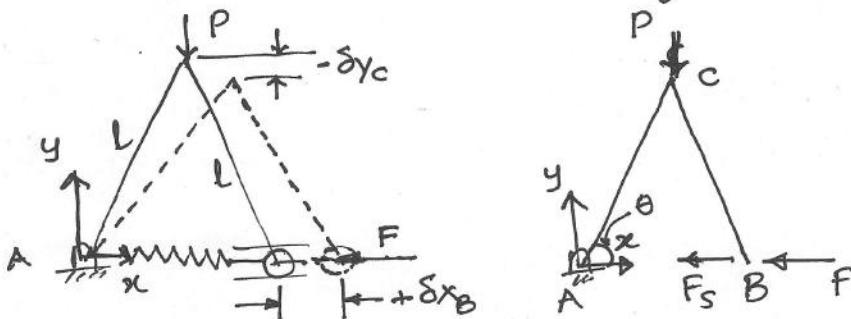
(2) If $F=0$, and P acts at C, $\therefore y_{D/C} = l \sin \theta; \delta y_D = l \cos \theta$

Eqn. (2) becomes

$$Pl \cos \theta - Q \cdot 2l \sin \theta = 0$$

$$\& Q = \frac{P}{2} \cot \theta, \text{ as in problem 1.}$$

Example: 4: Spring stiffness is K . Find the force required to maintain equilibrium. Free length of spring = a .



Taking the origin of the coord. system at A as shown,

$$\text{Force due to spring at } B \text{ (to left)} = K(2l \cos \theta - a)$$

Active forces are P, F_s and F . Taking virtual position as shown,

$$\delta U = P \cdot (-\delta y_C) - F \cdot (+\delta x_B) - F_s \cdot (+\delta x_B) = 0 \quad \dots (1)$$

$$y_C = l \sin \theta; \quad \delta y_C = l \cos \theta \cdot \delta \theta$$

$$x_B = 2l \cos \theta; \quad \delta x_B = -2l \sin \theta \cdot \delta \theta$$

Eqn.(1) becomes,

$$P \cdot (-l \cos \theta) - F_s \cdot (-2l \sin \theta) - F \cdot (-2l \sin \theta) = 0$$

$$\text{or } F \sin \theta = \frac{P}{2} \cos \theta - K(2l \cos \theta - a) (-l \sin \theta)$$

$$\text{or } F = \frac{P}{2} \cot \theta - K(2l \cos \theta - a) \quad \dots (2)$$

Note: 1. If there is no spring ($K=0$), then the expression for $F = \frac{P}{2} \cot \theta$ as in Example 1.

2. If F is absent, we can find the required spring force to obtain maintain a given angle ' θ ' as

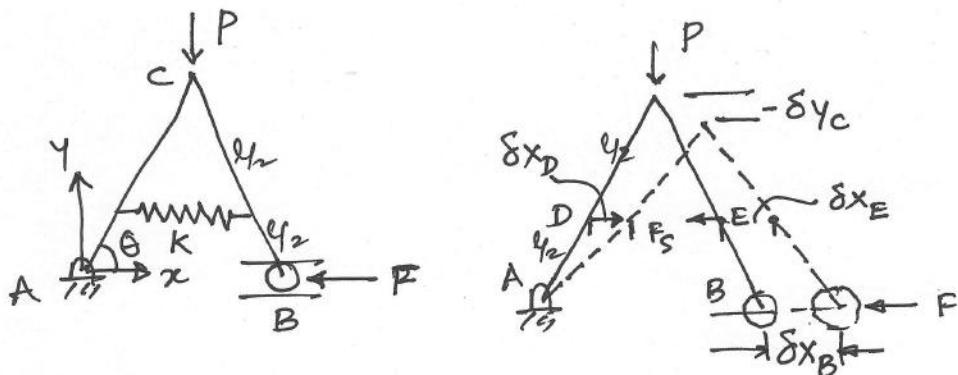
$$K = \frac{P \cot \theta}{2(2l \cos \theta - a)}$$

3. If P, K and R are known, Eqn. (2) can be written as

$$\frac{P}{2} \cot \theta = K(2l \cos \theta - a) + F$$

and θ reqd. for equilibrium is found. by trial and error.

Example 5: Same as above example, except that spring is above the supports.



As per the convention explained above,

$$\delta U = P(-\delta y_C) + F_s(\delta x_D) - F_s(\delta x_E) - F(\delta x_B) = 0.$$

Using $y_C = l \sin \theta; \delta y_C = l \cos \theta \cdot \delta \theta$

$$x_D = \frac{l}{2} \cos \theta; \delta x_D = -\frac{l}{2} \sin \theta \cdot \delta \theta$$

$$x_E = \frac{3l}{2} \cos \theta; \delta x_E = -\frac{3l}{2} \sin \theta \cdot \delta \theta$$

$$x_B = 2l \cos \theta; \delta x_B = -2l \sin \theta \cdot \delta \theta$$

$$\delta U = 0 \Rightarrow -Pl \cos \theta - F_s \cdot \frac{l}{2} \sin \theta + F_s \cdot \frac{3l}{2} \sin \theta + F \cdot 2l \sin \theta$$

$$\text{or } F = \frac{P}{2} \cot \theta - \frac{F_s}{2} = \frac{l}{2} (P \cot \theta - k(l \cos \theta - a))$$

In the above problem, if $P=200 \text{ N}$, $k=30 \text{ N/m}$, free length of spring = 1.5 m, and $F=100 \text{ N}$; find θ for equilibrium ($l=40 \text{ m}$)

Assuming a downward displacement (vertical) of point C,

$$\delta U = -200 \cdot \delta y_C + P_s \delta x_D - F_s \cdot \delta x_E + 100 \cdot \delta x_B = 0 \quad (1)$$

Using $P_s = 30(40 \cos \theta - 15)$

$$\delta y_C = 30 \sin \theta; \delta x_D = 30 \cos \theta \cdot \delta \theta; x_E = 50 \cos \theta; \delta x_E = -50 \sin \theta \cdot \delta \theta$$

$$x_B = 60 \cos \theta; \delta x_B = -60 \sin \theta \cdot \delta \theta.$$

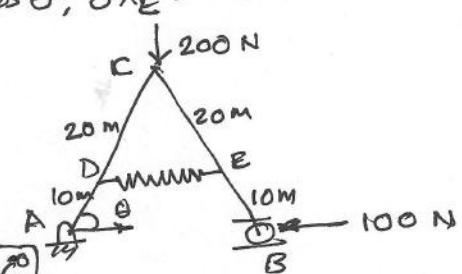
$$x_D = 10 \cos \theta; \delta x_D = -10 \sin \theta \cdot \delta \theta$$

Eqn. (1) leads to

$$8 \sin \theta - 4 \tan \theta = 1$$

'θ' is found by trial and error as

$$\theta = 53.6^\circ$$



Example 6: Find force F required to support a force P on the lift at angle θ .

(Note: As C moves up or down, A and D will be on the same vertical line. B and E are also on the same vertical line but move horizontally)

Active forces are P and F.

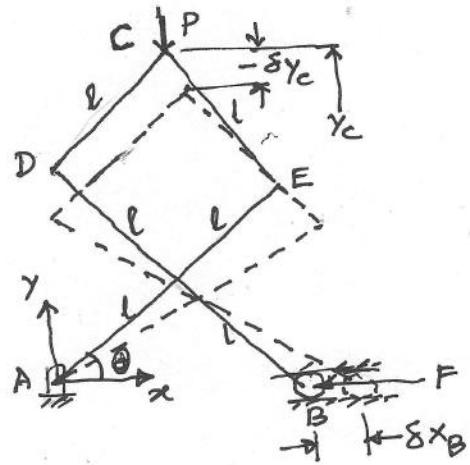
$$\delta U = +P(-\delta y_C) - F(+\delta x_B) = 0$$

$$y_C = 3l \sin \theta; \delta y_C = 3l \cos \theta \cdot \delta \theta$$

$$x_B = 2l \cos \theta; \delta x_B = -2l \sin \theta \cdot \delta \theta$$

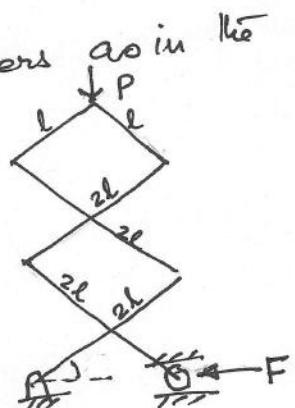
$$\text{Thus } \delta U = 0 \rightarrow -3Pl \cos \theta + 2Fl \sin \theta = 0$$

or
$$F = \frac{3P}{2} \cot \theta$$



Note: 1. If lift is of 6 members instead of 4 members as in the above problem, as shown (top links of length l and other links (4) of 2l)
We can find using the above procedure

$$F = \frac{5P}{2} \cot \theta.$$

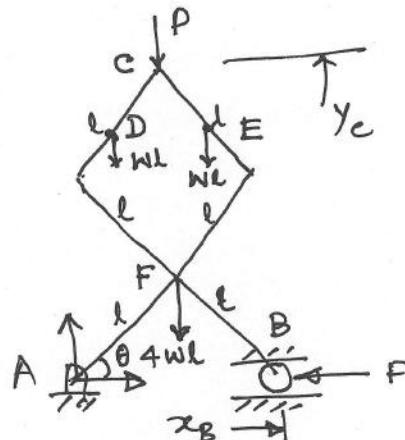


2. Use the same equation to find θ , the equilibrium position, if P and F are given.

Example: 7: If links are of wt. W N/m length in the above problem, find F required to maintain equilibrium at angle θ , when force P acts at the top.

Assuming virtual displacements as in the previous problem,

$$\begin{aligned}\delta U &= P(-\delta y_c) + 4Wl(-\delta y_F) \\ &\quad + Wl(-\delta y_D) + Wl(-\delta y_E) \\ &\quad - F \cdot \delta x_B \quad \rightarrow (1)\end{aligned}$$



We have

$$y_c = 3l \sin \theta; \delta y_c = 3l \cos \theta \cdot \delta \theta$$

$$y_D = y_E = \frac{5l}{2} \sin \theta; \delta y_D = \frac{5l}{2} \cos \theta \cdot \delta \theta = \delta y_E$$

$$y_F = l \sin \theta; \delta y_F = l \cos \theta \cdot \delta \theta; x_B = 2l \cos \theta; \delta x_B = -2l \sin \theta \cdot \delta \theta$$

Eqn(1) becomes

$$P(-3l \cos \theta) + 4Wl(-l \cos \theta) + 2Wl(-\frac{5l}{2} \cos \theta) - F(-2l \sin \theta) = 0$$

$$\rightarrow F = \left(\frac{3P + 9Wl}{2} \right) \cot \theta.$$

Problem: 8: For the system shown, find angle θ for equilibrium for a given force F , considering mass of each link as m .

Considering virtual displacement of F to right,

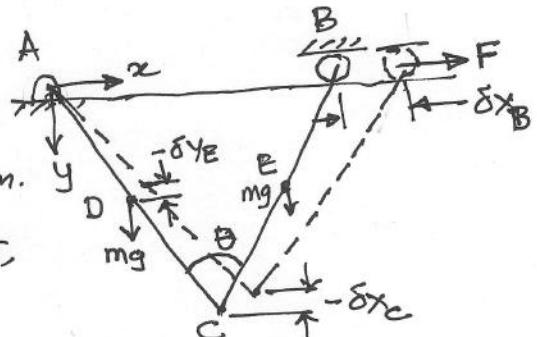
$$\delta U = F \cdot \delta x_B - 2mg(-\delta x_E) = 0$$

$$x_B = 2l \sin \theta/2; y_D, E = \frac{l}{2} \cos \theta/2$$

$$\delta U \Rightarrow F \cdot 2l \cos \theta/2 - 2mg \cdot \frac{l}{2} \sin \theta/2 = 0$$

$$\Rightarrow F = \frac{mg}{2} \tan \theta/2$$

$$\text{or } \theta = 2 \tan^{-1} \frac{2F}{mg}$$



$$\text{Length of } AC = BC = l$$

Problem : 9: For the system shown, K of spring = 2.7 N/m, and its free length is when $\theta = 45^\circ$. Find θ in equilibrium.

Active forces are force 270 N at D and spring force at A.

$$\text{When } \theta = 45^\circ, x = 600 \cos 45^\circ = 600/\sqrt{2}$$

Spring force is to right when $\theta < 45^\circ$

$$F_s = K(600 \cos \theta - 600/\sqrt{2})$$

$$\delta U = 0 = 270 \times (-\delta y_D) - F_s(\delta x_A)$$

$$y_D = 250 \sin \theta; \delta y_D = 250 \sin \theta \cdot \delta \theta$$

$$x_A = 600 \cos \theta; \delta x_A = -600 \sin \theta \cdot \delta \theta$$

$$\delta U = 0 \Rightarrow 270 \times 250 \cos \theta = 2.7 \times 600 (\cos \theta - 1/\sqrt{2}) \times 600 \sin \theta$$

$$\Rightarrow \frac{\sin \theta - \tan \theta}{\sqrt{2}} = 0.06944$$

$$\rightarrow \theta = 38^\circ \text{ - by trial and error. } (\theta = 15^\circ \text{ is also a solution})$$

Problem : 10 : Given $K = 1.8 \text{ kN/m}$, find θ for equilibrium.

M and F_s are the active forces.

$$\delta U = M \cdot \delta \theta - F_s \delta y = 0 \quad (1)$$

$$y = 175 \tan \theta; \delta y = 175 \sec^2 \theta \cdot \delta \theta$$

$$M = 160 \times 75 \text{ N-mm}$$

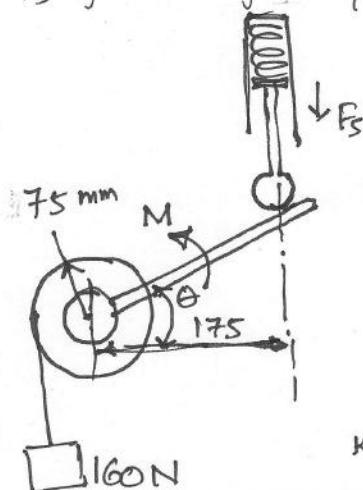
$$F_s \cdot \delta y = 1.8 \times (175)^2 \sec^2 \theta \cdot \tan \theta \cdot \delta \theta$$

Bqn. (1) gives,

$$160 \times 75 = \frac{1800 \times 175^2 \tan \theta \sec^2 \theta}{1000}$$

$$\frac{\tan \theta}{\cos^2 \theta} = \frac{160 \times 75}{1.8 \times (175)^2} = 0.2177$$

$$\boxed{\theta = 11.4^\circ} \text{ by trial and Error.}$$



$$K = 1.8 \text{ kN/m.} \\ = 1.8 \text{ N/mm}$$

$$F_s = K \cdot y \\ = \frac{1800 \times 175 \tan \theta}{1000}$$

Problem 11 : An automobile weighing W Newtons is lifted by a mechanism using hydraulic cylinder. Find the force reqd. by cylinder at an angle θ as shown. Link length is $2b$ and cylinder is hinged at mid point.

Let F be force due to cylinder.

Active Forces are W & F .

$$\delta U = -W \cdot \delta y_C + F \cdot \delta s$$

$$y_C = 2b \sin \theta + a$$

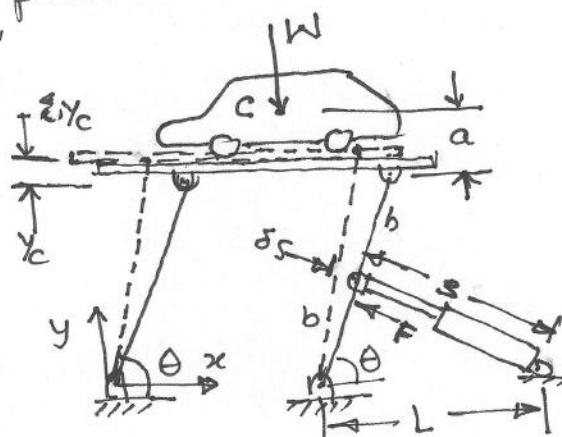
$$\delta y_C = 2b \cos \theta \cdot \delta \theta$$

$$s = \sqrt{b^2 + L^2 - 2bL \cos \theta}$$

$$\delta s = \frac{bL \sin \theta \cdot \delta \theta}{\sqrt{b^2 + L^2 - 2bL \cos \theta}}$$

$$F = \frac{W \delta y_C}{\delta s} = \frac{W \cdot 2b \cos \theta}{bL \sin \theta / (\sqrt{b^2 + L^2 - 2bL \cos \theta})}$$

$$\text{or } F = 2W \cot \theta \sqrt{1 + \frac{b^2}{L^2} - 2 \frac{b}{L} \cos \theta}$$



Problem 12: Find the couple M required for equilibrium at angle θ .

The active forces are W and M .

As M acts in the dirn. of increasing θ , virtual displacement is $+\delta \theta$.

$$\delta U = M \cdot \delta \theta - W \cdot (-\delta h)$$

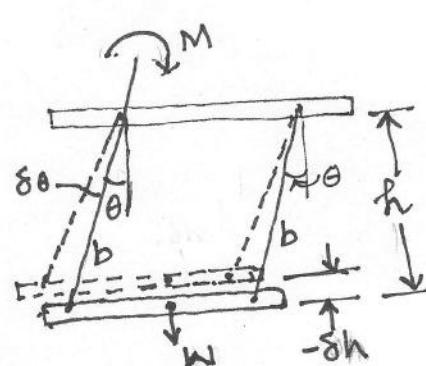
$$h = b \cos \theta; \delta h = -b \sin \theta \cdot \delta \theta$$

$$\text{Thus } \delta U = 0 \Rightarrow M \cdot \delta \theta = W \cdot \delta h = Wb \sin \theta \cdot \delta \theta$$

$$\text{or } M = Wb \sin \theta$$

or given W and couple M , the angle θ required for equilibrium can be found from

$$\theta = \sin^{-1} \frac{M}{Wb}$$



Example 13: For the system shown, find the force Q reqd. for equilibrium, given $L = 300 \text{ mm}$; $\theta = 50^\circ$ and $P = 100 \text{ N}$.

P and Q are the active forces.

$$\delta U = P \cdot \delta y_C - Q \cdot (-\delta x_A) = 0$$

We have

$$y_B = L \sin \theta; \delta y_B = L \cos \theta \cdot \delta \theta$$

$$\delta y_C = \frac{\delta y_B}{2} = \frac{L}{2} \cos \theta \cdot \delta \theta.$$

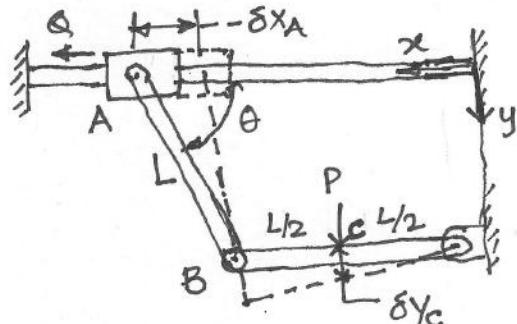
$$x_A = L + L \cos \theta; \delta x_A = -L \sin \theta \cdot \delta \theta$$

$$\text{Thus } \delta U = 0 \Rightarrow P \cdot \frac{L}{2} \cos \theta - Q \cdot L \sin \theta = 0$$

$$\therefore Q = \frac{P}{2} \cot \theta$$

Substituting $P = 100 \text{ N}$ & $\theta = 50^\circ$,

$$Q = 42 \text{ N}$$



Example 14: In a screw jack, force F is applied at joints A and C by a screw to lift a weight of 2000 N. Find F when angle $\theta = 30^\circ$.

Solution: Take origin as shown,

Y-axis passing through A. This avoids work due to F at A, as there is no motion of A where one F acts.

The active forces are P at B and F at C.

Assuming upward virtual motion of B,

$$\delta U = -P \cdot \delta y_B + F(-\delta x_C) = 0$$

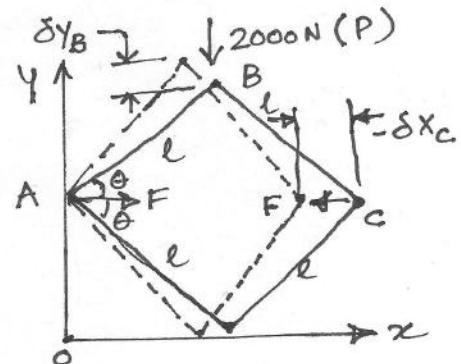
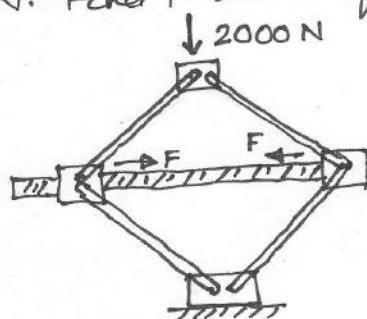
$$y_B = 2l \sin \theta; \delta y_B = 2l \cos \theta \cdot \delta \theta$$

$$x_C = 2l \cos \theta; \delta x_C = -2l \sin \theta \cdot \delta \theta$$

$$\text{Thus } \delta U = 0 \Rightarrow F = P \cot \theta$$

$$\text{or } F = 2000 \cot 30^\circ$$

$$= 3460 \text{ N}$$



Example: 15:

In a scissor lift mechanism, load W is lifted by turning a screw to change the distance between A and B.

Find the turning moment M required at an angle θ .

Lead of screw = L m/revolution.

Solution:

Work done by couple $M = M \cdot d\phi$, where $d\phi$ is the angle of rotation of the screw in the nut;

Thus the active forces are W and couple M .

Assuming an upward virtual displacement of W ,

$$\delta U = -W \cdot \delta y_c + M \cdot \delta \phi = 0.$$

$$y_c = 4b \sin \theta; \delta y_c = 4b \cos \theta \cdot \delta \theta$$

$$x_B = 2b \cos \theta; \delta x_B = -2b \sin \theta \cdot \delta \theta.$$

Relation between $\delta \phi$ and δx is

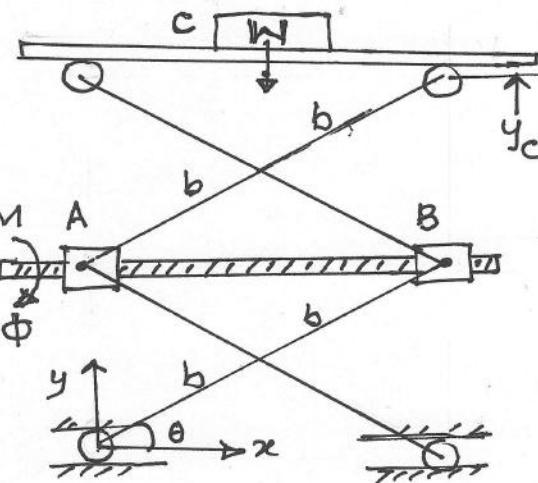
$$\delta x = -\frac{\delta \phi}{2\pi} \cdot L$$

$$\text{Hence } \delta x_B = -\frac{L}{2\pi} \delta \phi = -2b \sin \theta \cdot \delta \theta$$

$$\text{or } \delta \phi = 4\pi \frac{b}{L} \sin \theta \cdot \delta \theta.$$

$$\text{Thus } \delta U = 0 \Rightarrow -W \cdot 4b \cos \theta + M \cdot \frac{4\pi b}{L} \sin \theta = 0$$

$$\Rightarrow M = \frac{WL}{\pi} \cot \theta$$



Example 16: For the slider crank mechanism shown, given

$$l = 2r, M = 50 \text{ N.m}, \& \Phi = 35^\circ, r = 0.1 \text{ m}$$

find the force F required on slider to maintain equilibrium at $\theta = 30^\circ$.

Considering a virtual displacement as shown.

$$\delta U = M \cdot (-\delta\theta) - F \cdot \delta x_D = \Theta W \cdot \delta y_c = 0$$

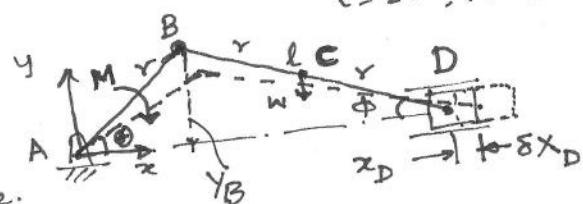
$$\text{or } M \delta\theta + F \delta x_D + W \delta y_c = 0$$

$$l = 2r; W = 35N$$

$$x_D = r \cos\theta + l \cos\Phi$$

x_D is in terms of θ and Φ

We need to get in terms of θ alone.
For this, we need to get geometric relation between θ and Φ .



$$r \sin\theta = l \sin\Phi$$

$$\cos\Phi = \sqrt{1 - \sin^2\Phi} = \sqrt{1 - \left(\frac{r \sin\theta}{l}\right)^2}$$

$$x_D = r \cos\theta + l \cos\Phi = r \cos\theta + \sqrt{l^2 - r^2 \sin^2\theta}$$

$$\delta x_D = -r \sin\theta \cdot \delta\theta - \frac{1}{2} \frac{2r^2 \sin\theta \cos\theta \cdot \delta\theta}{\sqrt{l^2 - r^2 \sin^2\theta}}$$

$$= -r \sin\theta \left(1 + \frac{r \cos\theta}{\sqrt{l^2 - r^2 \sin^2\theta}}\right) \delta\theta$$

$$y_c = \frac{1}{2} y_B = \frac{1}{2} r \sin\theta$$

$$\delta y_c = \frac{r}{2} \cos\theta \cdot \delta\theta$$

Eqn.(1) becomes

$$M \delta\theta - F r \sin\theta \left(1 + \frac{r \cos\theta}{\sqrt{l^2 - r^2 \sin^2\theta}}\right) \delta\theta + \frac{W r}{2} \cos\theta \cdot \delta\theta = 0$$

$$\text{or } F = \frac{M + \frac{W r \cos\theta}{2}}{r \sin\theta \left(1 + \frac{r \cos\theta}{\sqrt{l^2 - r^2 \sin^2\theta}}\right)}$$

Substituting given values

$$F = \frac{50 + \frac{35 \times 0.1}{2} \cos 30^\circ}{0.1 \times \sin 30 \left(1 + \frac{0.1 \cos 30}{\sqrt{0.2^2 - 0.1 \sin^2 30}}\right)}$$

$F = 712 \text{ N}$

Problem 17: Two uniform bars with rollers attached in the middle, are restricted to move in slots. Determine angle 'θ' when a couple M is applied.

Active forces are M, W and W.

$$\delta U = M \cdot (\delta \theta_2) - W \cdot \delta h_1 - W \cdot \delta h_2 = 0$$

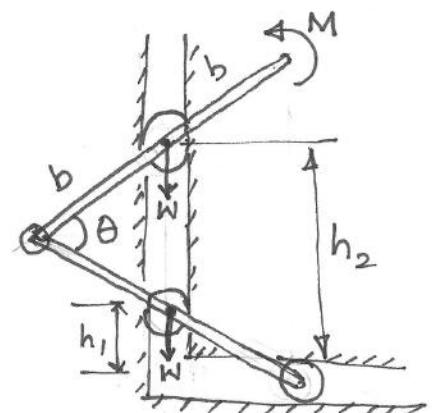
$$h_1 = b \sin \theta/2; \quad h_2 = 3b \sin \theta/2$$

$$\delta h_1 = b \cos \theta/2 \cdot \delta \theta/2; \quad \delta h_2 = 3b \cos \theta/2 \cdot \delta \theta/2.$$

$$\delta U \Rightarrow M - W b \cos(\theta/2) - 3W b \cos(\theta/2) = 0$$

$$M = 4 W b \cos(\theta/2)$$

$$\text{or } \boxed{\theta = 2 \cos^{-1} \frac{M}{4Wb}}$$



Problem 18:

A rod AB is hinged at one end A and at point C to another rod CD, which slides at the other end D. Find the force on the slider (F), when a 50N force is applied at B, at $\theta = 20^\circ$.

$$\delta U = -50 \cdot \delta y_B - F \cdot \delta x_D = 0 \quad \text{--- (1)}$$

$$y_B = 200 \sin \theta; \quad \delta y_B = 200 \cos \theta \cdot \delta \theta$$

$$x_D = 100 \cos \theta + 150 \cos \phi$$

$$\text{To relate } \theta \text{ and } \phi, \quad 100 \sin \theta = 150 \sin \phi$$

$$\cos \phi = \sqrt{1 - 4/9 \sin^2 \theta}$$

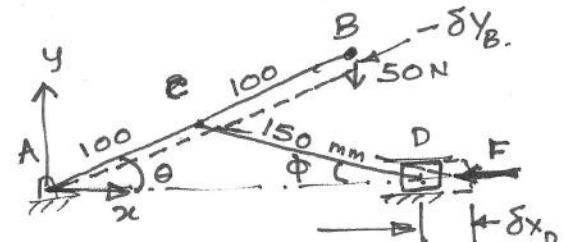
$$\text{Thus } x_D = 100 \cos \theta + 150 \times \sqrt{1 - 4/9 \sin^2 \theta}$$

$$\delta x_D = -100 \sin \theta \cdot \delta \theta - 150 \times \frac{8/9 \sin \theta \cos \theta}{\sqrt{1 - 4/9 \sin^2 \theta}}$$

$$\text{From Eq.(1), } F = \frac{50 \times 200 \cos \theta}{\left(100 + \frac{400/3 \cdot \cos \theta}{\sqrt{1 - 4/9 \sin^2 \theta}}\right) \sin \theta}.$$

Substituting $\theta = 20^\circ$,

$$\boxed{F = 120 \text{ N}}$$



Two Degrees of Freedom Systems

When work done is a function of two variables, $U(x, y)$, where x and y can vary independent of each other,

then $\delta U = \frac{\partial U}{\partial x_1} \cdot \delta x_1 + \frac{\partial U}{\partial x_2} \cdot \delta x_2$ is calculated as virtual work.

Each of the two terms is computed, treating the other variable as constant.

Example 19: Collars A & B can slide on rods as shown,

and are connected by a spring. Given $P = 600 \text{ N}$, $Q = 500 \text{ N}$, spring stiffness = 8000 N/m , unstretched length $L = 0.3 \text{ m}$; determine the values of x and y .

(i) Varying x , keeping y constant,

$$\delta U_1 = P \cdot \delta x - F_s \cdot \delta L \quad (1)$$

$$L^2 = x^2 + y^2 \quad (2)$$

$$2L \delta L = 2x \cdot \delta x \quad (y = \text{constant})$$

$$\therefore \delta L = \frac{x}{L} \cdot \delta x$$

$$\text{Eqn. (1)} \rightarrow P \cdot \delta x = F_s \cdot \frac{x}{L} \cdot \delta x$$

$$\text{or } 600 = F_s \cdot \frac{x}{L} \quad (3)$$

$$\text{By } \delta U_2 = 0 \quad (y \text{ varying, keeping } x \text{ constant})$$

$$\rightarrow 500 = F_s \cdot \frac{y}{L} \quad (4)$$

$$\text{We have } F_s = 6000(L - 0.3) = 6000L - 1800 \quad (5)$$

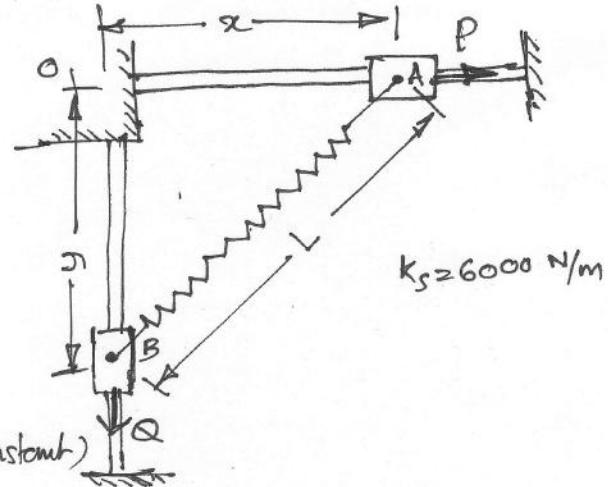
$$\text{From (3) \& (4), } F_s^2 (x^2 + y^2) = (500^2 + 600^2) L^2$$

$$\rightarrow F_s = 781 \text{ N}$$

$$\text{which from Eqn. (5)} \rightarrow L = 0.43 \text{ m}$$

Using Eqns. (3) and (4)

$x = 0.33 \text{ m}$
$y = 0.275 \text{ m}$



Example 20:

For the compound pendulum acted upon by a force P as shown, determine θ_1 and θ_2 .

(a) Keeping θ_1 constant and varying θ_2 ,

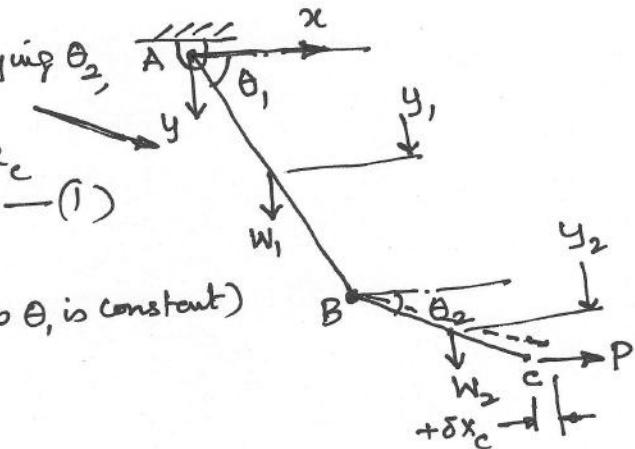
$$\delta U_2 = -W_2(-\delta y_2) + P \cdot \delta x_c \quad (1)$$

$$y_2 = a \sin \theta_1 + \frac{b}{2} \sin \theta_2$$

$$\delta y_2 = \frac{b}{2} \cos \theta_2 \cdot \delta \theta_2 \quad (\text{as } \theta_1 \text{ is constant})$$

$$x_c = a \cos \theta_1 + b \cos \theta_2$$

$$\delta x_c = -b \sin \theta_2 \cdot \delta \theta_2$$



$$\text{Eqn. (1)} \rightarrow \delta U_2 = 0 \rightarrow$$

$$W_2 \cdot \frac{b}{2} \cos \theta_2 - P \cdot b \sin \theta_2 = 0$$

$$\text{or } \boxed{\tan \theta_2 = \frac{W_2}{2P}}$$

(b) Keeping θ_2 constant and varying θ_1 ,

$$\delta U_1 = W_1 \cdot \delta y_1 + W_2 \cdot \delta y_2 + P \cdot \delta x_c = 0$$

$$y_1 = \frac{a}{2} \sin \theta_1; \delta y_1 = \frac{a}{2} \cos \theta_1 \cdot \delta \theta_1$$

$$y_2 = a \sin \theta_1 + \frac{b}{2} \sin \theta_2; \delta y_2 = a \cos \theta_1 \cdot \delta \theta_1$$

$$\& \delta x_c = -a \sin \theta_1 \cdot \delta \theta_1$$

$$\delta U_1 = 0 \rightarrow W_1 \cdot \frac{a}{2} \cos \theta_1 + W_2 \cdot a \cos \theta_1 - P a \sin \theta_1 = 0.$$

$$\boxed{\tan \theta_1 = \frac{W_1 + W_2}{P}}$$

Numerical: $W_1 = 100 \text{ N}$; $W_2 = 60 \text{ N}$; $a = 1 \text{ m}$; $b = 1 \text{ m}$; $P = 50 \text{ N}$.

$$\tan \theta_1 = \frac{110}{50}; \theta_1 = 65.5^\circ$$

$$\tan \theta_2 = \frac{60}{100}; \theta_2 = 31^\circ$$