

MVSR Engg. College, Nadargul, Hyderabad
Dept. of Mechanical Engg.

BE III (Mech) I & II Sections - Vibrations - Nov. '17. PAS
Assignment II and Practice problems:

Assignment 2 Problems 2, 3, 5, 7 . (Use $E = 200\text{GPa}$ for shaft material.)

Forced Vibrations

1. A vertical air compressor of 500 Kg mass is mounted on springs of $K = 1.96 \times 10^5 \text{ N/m}$ and dashpots of damping ratio 0.2 N/m/s. The reciprocating parts weigh 20 kg and stroke =0.2 m. Find amplitude of vertical motion and excitation force if compressor is operated at 200 rpm.
2. Static deflection of an automobile on its springs is 10 cms. Find the critical speed when travelling on a road which has undulation that can be approximated as a sine wave of 8 cms amplitude and wavelength of 16 cms. (damping ratio = 0.05) . Also find amplitude of vibration at 75 kmph. *(CO 302.5)*

Whirling Speed

3. A rotor 10 kg rotor is mounted midway on a 800 mm long 20 mm dia shaft. The CG of rotor is 0.1 mm away from its geometric centre. If system rotates at 50 rps, find the amplitude of vibration and dynamic load on bearings. *(CO 302.5)*
4. A vertical shaft 30 mm dia and 1 m long is mounted on long bearings and carries a pulley of 10 kg midway. CG of pulley is 0.5 mm away from shaft axis. Find (i) whirling speed (ii) bending stress in shaft at 2000 rpm.
5. Rotor of a turbocharger of 9 kg mass is keyed centrally to a shaft of 25 mm dia, 40 cm length between bearings. Shaft material density = 8 gm/cm³; and shaft may be treated as simply supported. Find (i). Critical speed (ii) Amplitude of rotation of rotor at 3200 rpm if eccentricity of rotor CG = 0.015 mm (iii) vibratory force transmitted to base. *(CO 302.5)*
6. A vertical steel shaft 15 mm dia is held in long bearings 1 m apart. It carries in its middle disc of 15 kg. Eccentricity of rotor is 0.3 mm. If the permissible tensile stress is 70 MN./m²., find (i) critical speed of shaft (b) range of speeds unsafe to run. Neglect mass of shaft.

Dunkerley Method

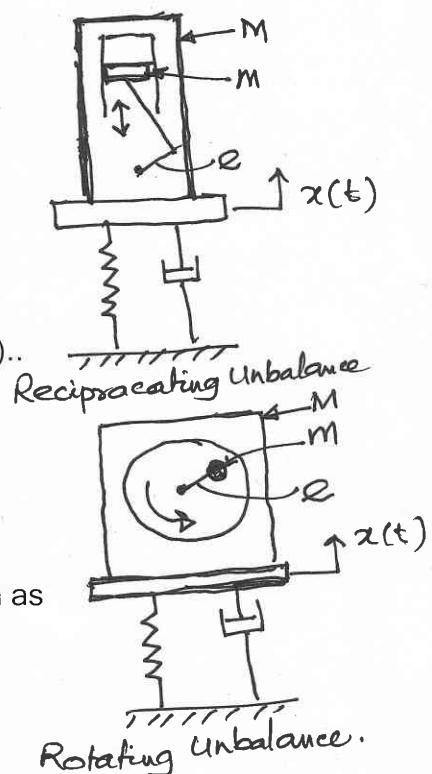
7. A shaft 1.5 m long supported at ends carries 2 wheels each of 60 kg as shown. Shaft is hollow with 75 mm x 40 mm dia. Density of shaft material is 7700 kg / m³. Find frequency of transverse vibration. *(CO 302.6)*

Rotating & Reciprocating Unbalance:

These two situations, when mounted on a spring and damper, lead to case of forced harmonic motion as described below.

Reciprocating Unbalance:

When a reciprocating machine, such as engine or compressor is mounted with cylinder along the x motion of the mass, the reciprocating mass of piston 'm' produces a harmonic force approximated as $m \omega^2 e$, where e is crank radius (= stroke/2)..



Rotating Unbalance:

This happens when rotating parts of mass 'm' have the CG at some eccentricity 'e'. from rotation axis. Considering the rotating unbalance, we can write the governing equation of motion as

$$(M-m) d^2x/dt^2 + m \cdot d^2/dt^2 (X + e \sin \omega t) = -kx - c \dot{x}$$

$$\text{or } M \ddot{x} + c \dot{x} + kx = m \omega^2 e \sin \omega t$$

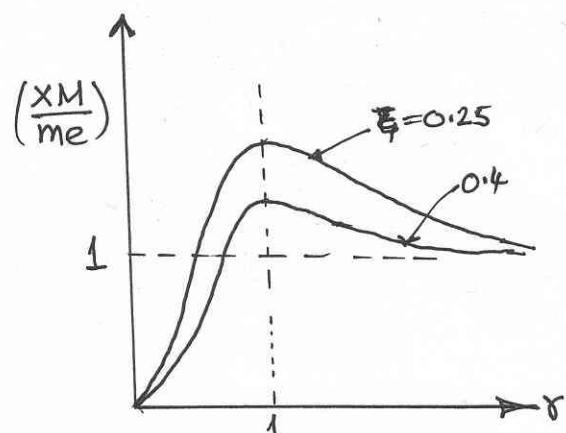
This is the dynamic equation of a forced vibration for which we get the amplitude of vibration as a function of ω given as

$$X = m \omega^2 e / \sqrt{(k - m \omega^2)^2 + (c \omega)^2}$$

$$\begin{aligned} &= \frac{m e \omega^2}{\sqrt{(k - m \omega^2)^2 + (c \omega)^2}} \\ &= \frac{m e \cdot \frac{M}{K} \cdot \omega^2}{\sqrt{\left(1 - \frac{m \omega^2}{K}\right)^2 + \left(\frac{c \omega}{K}\right)^2}} \\ \text{or } &\frac{X}{\left(\frac{m \omega}{M}\right)} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \end{aligned}$$

where $r = \omega/\omega_n$.

The corresponding magnitude ratio $X_m/m \omega$ is as shown in the figure. As $r > \infty$, $(X_m/m \omega)$ approaches 1.



Problem 1 : $M = 500 \text{ kg}$, $m = 20 \text{ kg}$, $k = 1.96 \times 10^5 \text{ N/m}$, $\xi = 0.2$

Stroke = 0.2 m , crank radius = $r = 0.2/2 = 0.1 \text{ m}$.

Exciting frequency = Compressor speed in rad/sec

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{1.96 \times 10^5}{500}} = 19.8 \text{ rad/sec}$$

$$\text{Frequency Ratio} = \frac{\omega}{\omega_n} = 1.057$$

$$\frac{x_{\max}}{(me/M)} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} = 2.456.$$

Hence amplitude at 200 rpm = $x_{\max} = \frac{20 \times 0.1}{500} \times 2.456 =$

Max. exciting force = $m\omega^2 = 20 \times (20.94)^2 \times 0.1 = [877 \text{ N}]$

Problem 2 : $\Delta = 10 \text{ cms}$; $\xi = 0.05$

This is a case of base motion $y = y \sin \omega t$

$$\text{where } \omega = \frac{\text{Speed}}{\text{Wavelength}} \times 2\pi \text{ rad/sec.}$$

$$\text{or Speed} = \frac{\omega}{2\pi} \times \text{Wavelength}$$

We have nat. frequency of automobile

$$\omega_n = \sqrt{\frac{g}{\Delta}} = \sqrt{\frac{9.81}{0.1}} = 9.9 \text{ r/s.} = 1.576 \text{ cps.}$$

When frequency of wheel harmonic motion is equal to the vehicle natural frequency, then the corresponding speed is critical.

(a) Hence critical speed = $\frac{\omega_n}{2\pi} \times \text{W.length} = \frac{9.9}{2\pi} \times 16 =$

$$= 25.21 \text{ m/s} = [90.76 \text{ Kmph}]$$

(b) $75 \text{ Kmph} = \frac{75000}{3600} \text{ m/s} = \frac{75000}{3600} \times \frac{1}{16} = 1.302 \text{ cycles/sec.}$
 $= 8.181 \text{ rad/sec.} = \omega$

$$\text{Frequency Ratio} = \frac{8.181}{9.9} = 0.8264.$$

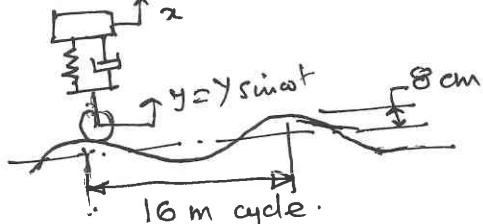
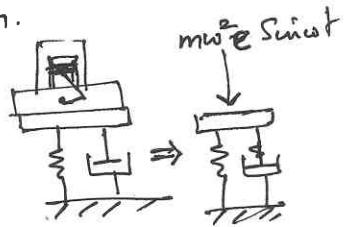
$$\text{For base motion we have } \frac{x}{x_0} = \frac{\sqrt{1 + (2\xi r)^2}}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

$$2\xi r = 2 \times 0.05 \times 0.8264 = 0.08264.$$

$$\frac{x}{x_0} = \frac{\sqrt{1 + 0.08264^2}}{\sqrt{(1 - 0.8264)^2 + 0.08264^2}} = 3.0623$$

$$x = x_0 \times 3.0623 = 8 \text{ cms} \times 3.0623 = [24.5 \text{ cms}]$$

Hence, vehicle body vibrates with 24.5 cms amplitude while travelling at 75 Kmph.



CRITICAL or WHIRLING Speed of shaft is the natural frequency of lateral vibration of shaft.

Let K = lateral stiffness of shaft, then
 frequency of lateral (or transverse) vibration $= \sqrt{\frac{K}{m}}$
 When shaft rotates, when the rotor mounted on the shaft has eccentricity 'e', then the centrifugal force $m\omega^2 e$, leading to a lateral deflection of the shaft ($= x$).

Let K = lateral stiffness of shaft

$\omega_n = \sqrt{K/m}$ is the natural frequency of lateral vibration.

Equating Centrifugal force = Lateral resisting force

$$m\omega^2 e = K \cdot x$$

$$m\omega^2(x+e) = K \cdot x \quad \text{or} \quad m\omega^2 e = (K - m\omega^2)x$$

$$\text{or } \frac{x}{e} = \frac{m\omega^2}{K - m\omega^2} = \frac{(m\omega^2/k)}{1 - m\omega^2/k} = \frac{(\omega/\omega_n)^2}{1 - (\omega/\omega_n)^2} \quad \text{where } \omega_n = \sqrt{k/m}$$

$$\text{Thus } \frac{x}{e} = \frac{r^2}{1-r^2}$$

As $r > 1$, x , the deflection of the shaft takes on very high values, and shaft may even break. Hence $r=1$ or $\omega=\omega_n$ is called the critical speed.

Therefore the natural frequency of lateral vibration is called critical or whirling speed.

Also, the maximum lateral force, or bending force on shaft is given as $F = Kx = m\omega^2 e$.

Problem 3: Lateral deflection $\delta = \frac{Wl^3}{48EI} = \frac{10 \times 9.81 \times 0.8^3}{48 \times 2 \times 10^{11} \times \frac{\pi}{64} \times 6.02} \times 4$

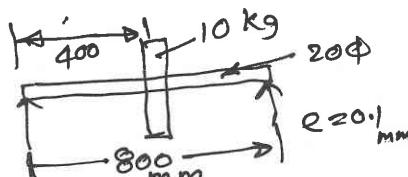
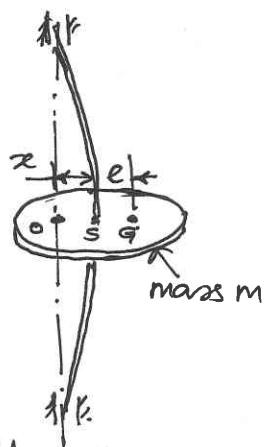
$$W_n = \sqrt{\frac{g}{l}} = 121.4 \text{ rad/sec} = 6.66 \times 10^{-4} \text{ m}$$

$$\text{Rotation speed } (\omega) = 50 \text{ rps} = 50 \times 2\pi = 314.2 \text{ rad/sec}$$

$$r = \omega/\omega_n = 314.2/121.4 = 2.588$$

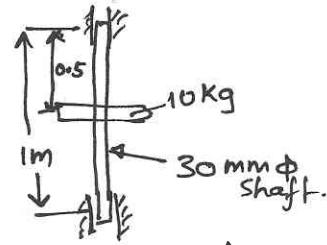
$$(a) \text{Amplitude of Lateral displacement } x_{(2)} = e \frac{r^2}{1-r^2} = -1.1755 \times 10^{-4} \text{ m}$$

$$(b) \text{Dyn. Load on bearings} = K \cdot x = \left(\frac{K}{8}\right) \cdot x \\ = \frac{10 \times 9.81}{6.66 \times 10^{-4}} \times 1.1755 \times 10^{-4} = 17.31 \text{ N}$$



Problem 4: Being Long bearings, we treat

the ends as fixed; for which case
deflection $\delta = \frac{WL^3}{192EI} = \frac{10 \times 9.81 \times 1^3}{192 \times 2 \times 10^{11} \times \pi/64 \times 0.03^4}$
 $= 6.425 \times 10^{-5} \text{ m}$



(a) $\omega_n = \sqrt{g/l} = 390.75 \text{ rad/s} (= 3731 \text{ rpm})$ is the Critical Speed or whirling speed.

(b) Lateral deflection $x = e \cdot \frac{r^2}{1-r^2}$

Actual speed of shaft = 2000 rpm.

$$r = \frac{\omega}{\omega_n} = \frac{2000}{3731} = 0.536.$$

$$x = 0.3 \times r^2 / (1-r^2) = 0.201 \text{ mm} = 0.201 \times 10^{-3} \text{ m.}$$

$$\text{Bending force} = k \cdot x = \frac{W}{8} \cdot x = \left(\frac{10 \times 9.81}{6.425 \times 10^{-5}} \right) \times 0.2 \times 10^{-3} = 307 \text{ N.}$$

Bending Moment for fixed beam = $Fl/8 = 307 \times 1/8 \text{ N.m.}$

From flexure formula, $\frac{M}{I} = \frac{f}{y}$ or $f = \frac{M}{I} \cdot y$

$$\text{where } M = 307/8; y = d/2 = 0.015 \text{ m} \quad & I = \pi/64 \times 0.03^4$$

gives Bending stress 'f' = 14.48 MPa at 2000 rpm.

Problem 5: For u.d.l (as self weight of beam), deflection at the

centre of a simply supported beam = $\frac{5}{384} \frac{wl^4}{EI}$

where $w = \text{weight/unit length} = \frac{\pi}{4} \times 2.5^2 \times 1 \times 8 \times 9.81 \times 10^3 \text{ gm/cm}^3$
 $= (\frac{\pi}{4} \times 2.5^2 \times 1 \times 8) / 1000 \times 9.81 \times 100 = 38.52 \text{ N/m.}$

$$\delta = \text{Total deflection} = \text{due to } 9 \text{ kg mass} + \text{Self Weight} = \frac{wl^3}{48EI} + \frac{5}{384} \frac{wl^4}{EI}$$

$$= \frac{l^3}{EI} \left(\frac{w}{48} + \frac{5}{384} \frac{wl}{E} \right) = \frac{0.4^3}{2 \times 10^{11} \times (\frac{\pi \times 0.025^4}{64})} \left(\frac{9 \times 9.81}{48} + \frac{5}{384} \times 38.52 \times 0.4 \right)$$

$$= 3.4 \times 10^{-5} \text{ m}$$

(a) Critical speed = $\omega_n = \sqrt{g/l} = 537 \text{ rad/sec} = 5128 \text{ rpm.}$

• $r = \frac{\text{Shaft Speed}}{\omega_n} = \frac{3200}{5128} = 0.624. \text{ (frequency ratio = r)}$

• Amplitude of vibration $x = e \frac{r^2}{1-r^2} = 0.00957 \text{ mm}$ ($e = 0.015 \text{ mm}$)

(b) Dynamic force = Lateral Stiffness \times displacement
on bearings = $K \cdot x$

Where $K = \text{Total Load / Stiffness}$

We can obtain 'K' as Load/deflection, where ~~Load~~ Load is the concentrated load and given and the equivalent concentrated load of u.d.l..

Equivalent wt. W_e of shaft causing deflection as a concentrated load is obtained from

$$\frac{W_e l^3}{48 EI} = \frac{5}{384} \frac{WL^4}{EI} \rightarrow W_e = \frac{5}{8} WL$$

Hence equivalent stiffness K = $\frac{\text{Total Load}}{\delta} = \frac{W + W_e}{\delta}$

$$= [9 \times 9.81 + (5/8 \times 38.52 \times 0.4)] / 3.4 \times 10^{-5}$$

$$= 2879852 \text{ N/m.}$$

(c) Vibratory force transmitted to the two bearings

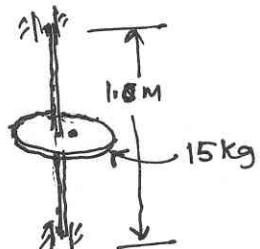
$$= K \cdot x = 2879852 \times 0.00957 \times 10^{-3}$$

$$= \boxed{27.56 \text{ N}}$$

Problem 6: As it is mentioned 'Long' bearings, we take the b. rod as fixed supported.

Hence deflection $\delta = \frac{Wl^3}{192EI} = 1.542 \times 10^{-3}$

$$\omega_n = \sqrt{\frac{W}{I}} = \sqrt{\frac{9.81}{1.542 \times 10^{-3}}} = 80 \text{ rad/sec} (= 762 \text{ rpm})$$



Using flexure formula, $\frac{M}{I} = f/y$,

$$\text{With } f_{max} = 70 \times 10^6, y = \underline{0.015/2} \text{ (d/2)} \leftarrow M = \frac{fyl}{8}$$

$$M = f \cdot I/y = \frac{70 \times 10^6 \times \pi/64 \times (0.015)^4}{(0.015/2)} = 23.19 \text{ N.m}$$

$$23.19 = Wl/8 \text{ gives } W_e = 23.19 \times 8 = 185 \text{ N}$$

$$\delta \text{ due to this load} = \frac{W_e l^3}{192 EI} = 1.944 \times 10^{-3} \text{ m}$$

$$\text{From the equation } \pm \frac{x}{e} = \frac{r^2}{1-r^2} \text{ or } \frac{1.944 \times 10^{-3}}{0.003 \times 10^{-3}} = 6.48$$

$$\text{Using } \frac{r^2}{(1-r^2)} = +6.48 \rightarrow r = 0.93 = \omega/\omega_n = \omega/80$$

$$\frac{r^2}{(1-r^2)} = -6.48 \rightarrow r = 1.087 = \omega/80$$

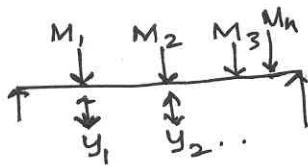
$$\omega_1 = 0.93 \times 80 = 74.4 \text{ rad/sec} (= 710 \text{ rpm})$$

$$\omega_2 = 1.087 \times 80 = 86.96 \text{ rad/sec} (= 830 \text{ rpm})$$

Hence it is unsafe to run the shaft between 710 rpm and 830 rpm as the bending stress exceed the permissible 70 MN/m^2 .

RALEIGH'S METHOD

(i) Concentrated Loads.



For multi d.o.f systems, we can obtain the fundamental frequency (lowest natural)

by equating the total PE to KE due to all loads, when the points of application go thro a harmonic vibration at certain frequency. $KE = PE$ may be written as

$$\frac{1}{2} \sum_{i=1}^n m_i (y_i w_i)^2 = \frac{1}{2} \sum_{i=1}^n m_i y_i^2$$

$$\text{or } w_n^2 = \frac{g \sum m_i y_i}{\sum m_i y_i^2} = \frac{g(m_1 y_1 + m_2 y_2 + \dots)}{(m_1 y_1^2 + m_2 y_2^2 + \dots)}$$

Here if y_i is the deflection at station i due to all loads, i.e. m_1, m_2, \dots, m_n . We can write

$$y_1 = \delta_{11} w_1 + \delta_{12} w_2 + \delta_{13} w_3 + \dots$$

$$y_2 = \delta_{21} w_1 + \delta_{22} w_2 + \delta_{23} w_3 + \dots$$

in which δ_{ij} is the deflection at i due to unit load at j , and is called the 'influence coefficient'.

For a simply supported beam, we can arrive at δ_{ij}

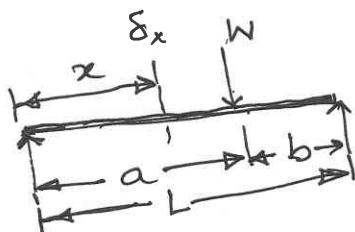
using the formula,

$$\delta = \frac{w(L-a)x(L^2 - a^2 - b^2)}{6EI L}$$

Due to unit Load (i.e. $w=1$)

$$\delta_{ij} = \frac{(L-a)x(L^2 - a^2 - b^2)}{6EI L}$$

Where station i is at distance ' x ', and j is at distance ' a ' from the left support.



Raleigh's Method for u.d.l.

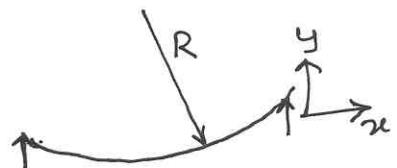
Raleigh's method can be applied for u.d.l or considering the weight of the beam using $KE = PE$.

Considering a uniform beam of length L and having ρ as mass/unit length,

$$\text{We have } P.E = \frac{1}{2} \int_0^L M \cdot d\theta$$

From theory of bending,

$$\frac{M}{EI} = \frac{1}{R} = \frac{d\theta}{dx} = \frac{d^2y}{dx^2}$$



$$\text{Then } P.E = \frac{1}{2} \int_0^L M \cdot \frac{dx}{R} = \frac{1}{2} \int_0^L M \cdot \frac{M}{EI} \cdot dx$$

$$= \frac{1}{2} EI \int_0^L \left(\frac{M}{EI}\right)^2 dx = \frac{1}{2} EI \int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$K.E = \frac{1}{2} \int_0^L (y \omega_n)^2 \rho \cdot dx = \frac{1}{2} \rho \omega_n^2 \int_0^L y^2 dx$$

Equating KE & PE,

$$\omega_n^2 = \frac{EI}{\rho} \cdot \frac{\int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx}{\int_0^L y^2 dx} \quad \text{--- (A)}$$

An approximate shape of $y(x)$ gives a reasonable value of ω_n .

For example, for a simply supported beam,

$$\text{let } y = y \sin \frac{\pi x}{L}, \quad (y=0 \text{ at } x=0 \text{ and } L)$$

Substituting in Eqn.(A) above,

$$\omega_n^2 = \frac{EI}{\rho} \cdot \frac{\int_0^L -\frac{\pi^2}{L^2} y \sin^2 \frac{\pi x}{L} dx}{\int_0^L y^2 dx} = \frac{EI}{\rho} \frac{\pi^4}{L^4}$$

$$\text{or } \omega_n = \frac{\pi^2}{L^2} \sqrt{\frac{EI}{\rho}} \quad \text{Which is accurate.}$$

Note: An accurate shape $y(x)$ assumed gives the lowest natural frequency, since other shapes mean that there are constraints, and constraints make the beam rigid, leading to higher frequency.

Effect of Mass of the beam can be included in Dunkerley's equation i.e. $\frac{1}{\omega_n^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \dots + \frac{1}{\omega_b^2}$

where ω_b is the natural frequency due to beam mass.

As seen earlier for simply supported beam,

$$\text{we have } \omega_b = \frac{\pi^2}{L^2} \sqrt{\frac{EI}{P}} \quad \text{--- (1)}$$

Also knowing deflection at centre due to u.d.l or own mass, for a simply supported beam is given by

$$\delta_b = \frac{5}{384} \frac{wl^4}{EI} \quad \text{--- (2)} \quad (\omega = Pg)$$

$$\text{Thus } \omega_b^2 = \frac{\pi^4}{L^4} \frac{EI}{P} = \pi^4 \times \frac{5}{384} \frac{g}{\delta_b} = \frac{g}{(\delta_b / 1.27)}$$

$$(\text{on substituting } \frac{EI}{PL^4} = \frac{5g}{384 \times \delta_b} \text{ from Eqn.2})$$

Hence for multi d.o.f system on a simply supported beam, including beam mass, Dunkerley's eqn. becomes

$$\frac{1}{\omega_n^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \dots + \frac{1}{\omega_b^2} = \frac{\delta_1}{g} + \frac{\delta_2}{g} + \dots + \frac{\delta_b / 1.27}{g}$$

$$\text{or } \omega_n = \sqrt{\frac{g}{\delta_1 + \delta_2 + \dots + \frac{\delta_b}{1.27}}} \quad (\text{where } \delta_b \text{ is the deflection at centre due to its own mass})$$

This formula is used in problem (7).

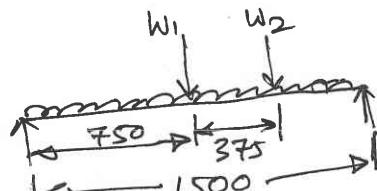
Problem 7:

$$w_1 = w_2 = 60g \quad ; \quad E = 2 \times 10^{11} \text{ N/m}^2$$

$$I_{\text{shaft}} = \frac{\pi}{64} (D^4 - d^4)$$

$$= \frac{\pi}{64} \times (7.5^4 - 4^4) \times 10^{-8} = 1.4275 \times 10^{-6} \text{ m}^4$$

$$\text{Weight per unit length} = \frac{\pi}{4} (4.5^2 - 4^2) \times 1 \times 7700 \times 10^4 \times g \\ = 24.341g = 238.79 \text{ N/m.}$$



$$\text{Deflection at Load 1} = \delta_1 = \frac{W a^2 b^2}{3 E I L}$$

$$= \frac{600 \times 9.81 \times 0.75^2 \times 0.75^2}{3 \times 2 \times 10^9 \times 1.4275 \times 10^{-6} \times 1.5} = 1.4496 \times 10^{-4} \text{ m.}$$

$$\delta_2 = \frac{600 \times 9.81 \times 1.125^2 \times 0.275^2}{3 \times 2 \times 10^9 \times 1.4275 \times 10^{-6} \times 1.5} = 0.8154 \times 10^{-4} \text{ m.}$$

$$\delta_{\text{shaft}} = \frac{5}{384} \frac{w L^4}{E I} = \frac{5}{384} \times \frac{238.8 \times 1.5^4}{2 \times 10^9 \times 1.4275 \times 10^{-6}} = 0.5513 \times 10^{-4} \text{ m.}$$

from Dunkerley's method

$$\omega_n = \sqrt{\frac{g}{\delta_1 + \delta_2 + \frac{\delta_s}{1.27}}} = \sqrt{\frac{9.81}{(1.4496 + 0.8154 + \frac{0.5513}{1.27}) \times 10^{-4}}}$$

$$\boxed{\omega_n = 190.64 \text{ rad/sec.}}$$

- without considering the shaft weight, we have

$$\omega_n = \sqrt{\frac{g}{\delta_1 + \delta_2}} = 208.1 \text{ rad/sec.}$$

————— x —————

Comparison of Dunkerley & Raleigh's Methods

Example: Find the natural frequency of a shaft which has rotors of 800 N and 1000 N mounted as shown.

Shaft dia = 20 mm. $E = 2 \times 10^{11} \text{ N/m}$.

$$I = \frac{\pi}{64} \times 2^4 \times 10^{-8} = 0.7854 \times 10^{-8} \text{ m}^4.$$

$$\delta_{11} = \frac{\alpha^2 b^2}{3EI} = \frac{0.3^2 \times 0.3^2}{3 \times 2 \times 10^{11} \times 0.7854 \times 10^{-8} \times 0.6} =$$

$$= 2.865 \times 10^{-6} \text{ m/N}$$

$$\delta_{11} = w_1 \cdot \delta_{11} = 800 \times \delta_{11} = 2.2918 \times 10^{-3} \text{ m.}$$

$$\delta_{22} = \frac{\alpha^2 b^2}{3EI} = \frac{0.5^2 \times 0.1^2}{3 \times 2 \times 10^{11} \times 0.7854 \times 10^{-8} \times 0.6} = 0.884 \times 10^{-6} \text{ N/m.}$$

$$\delta_2 = w_2 \times \delta_{22} = 1000 \times \delta_{22} = 0.884 \times 10^{-3} \text{ m.}$$

Using Dunkerley's Eqn. $\omega_n = \sqrt{\frac{g}{\delta_{11} + \delta_2}} = \sqrt{\frac{9.81}{(2.2918 + 0.884) \times 10^{-3}}} = 55.58 \text{ rad/s}$

→ Now to use Raleigh's Method, we need to calculate the deflection at each point due to both loads.

$$\therefore \delta_1 = \delta_{11} \times w_1 + \delta_{12} \times w_2 \quad \& \quad \delta_2 = \delta_{21} \times w_1 + \delta_{22} \times w_2 \quad \text{--- (A)}$$

$$\delta_{12} = \frac{bx}{6EI} (L^2 - x^2 - b^2) \quad (x = 0.3, b = 0.1)$$

$$= \frac{0.1 \times 0.3 (0.6^2 - 0.3^2 - 0.1^2)}{6 \times 2 \times 10^{11} \times 0.7854 \times 10^{-8} \times 0.6} = 1.379 \times 10^{-6} \text{ m/N}$$

$$\delta_{12} = \delta_{21}.$$

Substituting in equations (A), we get

$$\delta_1 = \delta_{11} \times w_1 + \delta_{12} \times w_2 = 3.6708 \times 10^{-3} \text{ m.}$$

$$\delta_2 = \delta_{21} \times w_1 + \delta_{22} \times w_2 = 1.9872 \times 10^{-3} \text{ m.}$$

Natural frequency from Raleigh's equation is

$$\omega_n^2 = \frac{g(m_1 \delta_1 + m_2 \delta_2)}{m_1 \delta_1^2 + m_2 \delta_2^2}$$

$$= 3280$$

$$(\text{using } m_1 = \frac{800}{g} \text{ & } m_2 = \frac{1000}{g})$$

$$\boxed{\omega_n = 57.27 \text{ rad/sec}}$$

