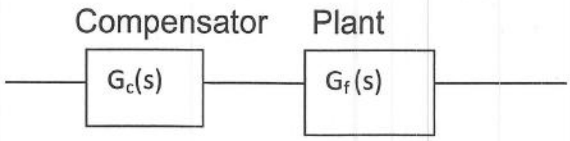
COMPENSATORS

To improve the performance of a given plant or system $G_f(s)$ it may be necessary to use a compensator or controller $G_c(s)$.



The requirements of a plant may be expressed in terms of
(a) settling time (b) damping ratio (c) peak overshoot --- in time domain
(d) phase margin or (e) gain margin ----- in frequency domain.

If the given plant does not satisfy 2 or 3 conditions specified from the above, a compensator circuit may be necessary to modify the overall plant characteristics (i.e of $G_c(s)$ $G_f(s)$) to satisfy the given specifications.

The controllers used are typically (other than P, I, D combination controllers)

Lead compensator – (to speed up transient response, margin of stability and improve error constant in a limited way)

Lag compensator – (to improve error constant or steady-state behavior – while retaining transient response)

Lead – Lag compensator – (A combination of the above two i.e. to improve steady state as well as transient).

<u>Lead Compensator</u>:- is so called since it adds phase angle so that the phase curve of the combined system is lifted above the -180^0 line so much as to provide the necessary phase margin. A compensator

$\frac{Ts+1}{\alpha\tau s+1}$ gives a phase angle

$\Phi_c = tan^{-1}\omega\tau - tan^{-1} \propto \omega\tau.$

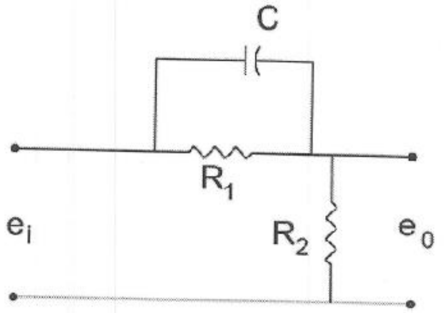
If $\propto < 1$, then Φ_c is positive. (or lead).

In pole - zero form, it is given as

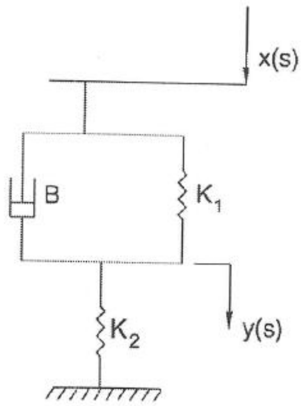
$G_c = \frac{s+Zc}{s+Pc} = \frac{s+\frac{1}{\tau}}{s+\frac{1}{\alpha\tau}}$ with $\alpha = Z_c/P_c$

$= \alpha \frac{(\tau s+1)}{(\alpha\tau s+1)}$ also gives a lead angle.

Lead compensator can be realized with electrical or mechanical components



Mechanical



$\frac{e_o}{e_i} = \frac{R2}{R2+\frac{R1}{1+R1Cs}} = \frac{R2}{R1+R2}\frac{(1+R1Cs)}{1+\frac{R2}{R1+R2}R1Cs}$ = $G_c(s)$

Letting τ = R_1 C; and $\alpha = R_2/(R_1+R_2) < 1$

$G_c(s) = \alpha\frac{(\tau s+1)}{(\alpha\tau s+1)}$--- (1)

with $\tau = B_1 / K_1$; $\alpha = K_1/(K_1+K_2)$

$Y(s)/X(s) = \alpha(\tau s+1)/(\alpha\tau s+1)$

The phase difference for $G_c(s)$ given by Eqn. (1) is $\varnothing_c = tan^{-1}\omega\tau - tan^{-1}\propto\omega\tau$

or $\tan\Phi = \omega\tau(1-\alpha)/(1+\alpha\omega^2\tau^2)$

$d\Phi/d\omega = 0$ gives the frequency for maximum phase difference as

$\omega_m = \frac{1}{\tau\sqrt{\alpha}}$ - which is the geometric mean of corner frequencies.

The max. phase lead is given as $\tan\Phi_m = \frac{1-\alpha}{2\sqrt{\alpha}}$ or $\sin\Phi_m = \frac{1-\alpha}{1+\alpha}$

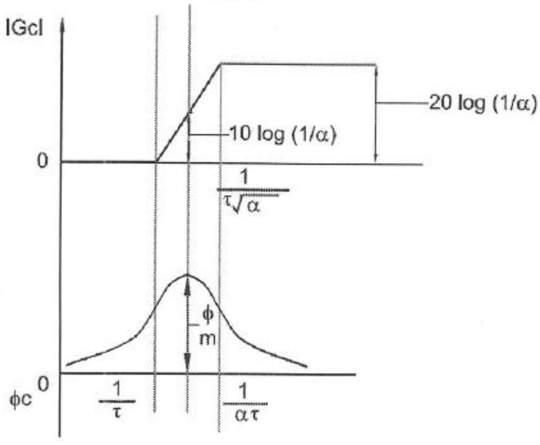
or $\alpha = \frac{1-\sin\Phi_m}{1+\sin\Phi_m}$

The maximum gain is 20log $1/\alpha$ at high frequency,

and the gain at ω_m *is* $G_c(\omega_m) = \frac{1}{\sqrt{\alpha}}$

or 20 log $G_c(\omega_m)$ =10 log $\frac{1}{\alpha}$, db

is the gain due to compensator at the mean frequency, where phase contribution is maximum.



Lead Compensator Frequency Response

<u>Design of Compensator for a system</u>

Design can be carried out

(i) in frequency domain or (ii) in time domain using Root-locus

Both methods are explained – for lead &lag compensator cases.

Frequency domain approach

Lead Compensator : Design Procedure - Steps

i) For the given plant – determine K_v or K_a based on specification

ii) Draw Bode Plot and check whether adequate Φ is provided

If ξ is specified, find the required phase margin from approx relation

$\xi = 0.01\Phi_{margin}$ or $\Phi_{margin} = 100\ \xi$ - becomes the specified phase margin

iii) Determine the phase lead to be introduced by controller as $\Phi_m = \Phi_s - \Phi_{uc} + \in$

Where Φ_s is the specified phase margin, Φ_{uc} - phase margin of uncompensated system, and $\in$ to account for shift of cross-over frequency.

iv) Determine α from $\alpha = \frac{1-\sin\Phi_m}{1+\sin\Phi_m}$

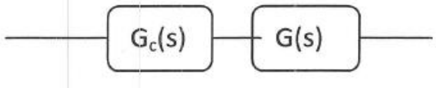
(If $Ø_m$ is more than 60^0, it is desirable to provide two identical networks each providing ½ the phase)

v) Draw a line 10 log1/α below 0, db on the uncompensated Bode plot and locate the corresponding frequency $\omega_m = 1/\tau\sqrt{\alpha}$;

vi) Determine τ, as α is known.

vii) Thus $G_c(s) = \frac{\tau s+1}{\alpha\tau s+1}$

Compensated system is $G_c(s)\ G_f(s)$



Example: Design a lead compensater for a system

$$G(s)\,H(s) = \frac{K}{s(s+1)(s+2)}$$

to satisfy $K_V = 10$ and damping ratio $\xi = 0.2$

Step 1: $K_V = \lim_{s\to 0} s\,G(s)H(s) = \frac{K}{2} = 10$; Hence $K = 20$.

Thus $G(s)\,H(s) = \dfrac{20}{s(s+1)(s+2)}$

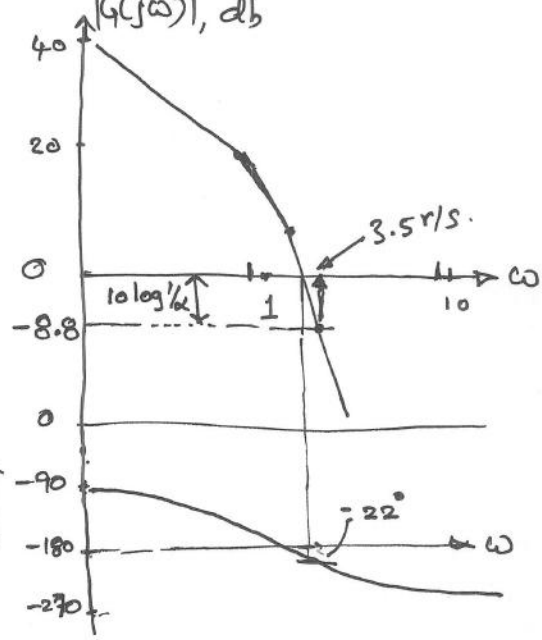
Also $\phi_M = 100\times\xi = 100\times 0.2 = 20^\circ$

Step 2: Draw Bode plot

(AR curve by Asym. approximation & ϕ from formula $\phi = -90 - \tan^{-1}\omega - \tan^{-1}\omega/2$

$\phi_{\omega=1} = -161^\circ$;

$\phi_{\omega=2} = -198^\circ$



Step 3: Existing phase margin $= -22^\circ$

Phase ~~margin~~ angle to be provided by

Compensator $= \phi_{M(reqd)} - \phi_{M_{existing}} + \epsilon$

or $\phi_m = 20 - (-22) + 8^\circ = 50^\circ$

Step 4:

$$\alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m} = \frac{1-\sin 50^\circ}{1+\sin 50} = 0.132$$

Step 5: $-10\log 1/\alpha = -8.8$ db cuts the gain line

at $\omega = 3.5$ r/s, which is made the mean Corner frequency

Hence $\omega = 3.5 = \dfrac{1}{T\sqrt{\alpha}}$

$$T = \frac{1}{3.5\sqrt{0.132}} = 0.786.$$

$$\alpha T = 0.132 \times 0.786 = 0.104$$

Step 6:

The compensator (Lead) is

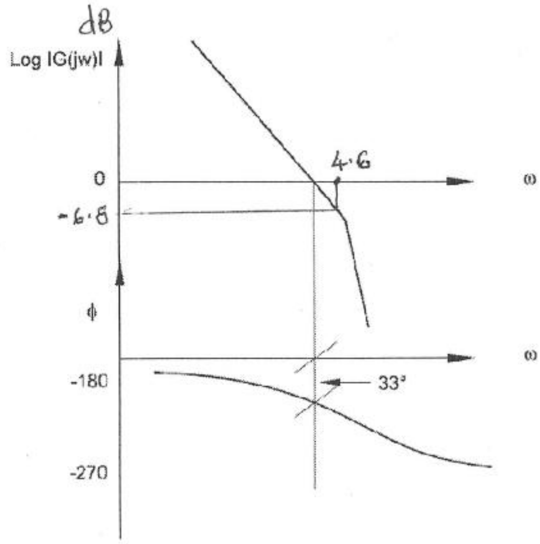
$$G_c(s) = \frac{1+Ts}{1+\alpha T s} = \frac{1+0.786s}{1+0.104s}$$

and T.F of Compensated plant $= G_c(s)G(s) = \left(\dfrac{1+0.786s}{1+0.104s}\right)\left(\dfrac{20}{s(s+1)(s+2)}\right)$

<u>Example</u> (from Nagrath & Gopal) - To design a compensator for a system $G(s) = \frac{K}{s^2(0.2s+1)}$ so that Acceleration Error Const. $K_a = 10$, and phase margin = 35^0

i) $K_a = s^2\, G(s)|_{s \to o} = K = 10$. Hence choose K= 10

ii) Draw Bode Plot for $G(s) = \frac{10}{s^2(0.2s+1)}$.



Phase margin is -33^0, system quite unstable.

Phase lead to be introduced by network

$= 35^0 - (-33^0) + \in$.

Chose $\in$ 12 to 14^0as gain slope in high.

Thus $\Phi_e = 35+33+14 = 82^0$.

Being high, two networks in tandem are required with $\Phi_m = 82/2 = 41^0$ each.

iii) $\alpha = (1\text{-}\sin 41^0)/(1+\sin 41^0) = 0.21$

iv) Gain at mean frequency = $10 \log(1/\alpha)$, = 6.8 dB, for each network.

v) Gain of UC system is -6.8 dB at $\omega = 4.6$ r/s, where Φ is maximum.

vi) Hence $\omega_m = 1/\tau\sqrt{\alpha}$

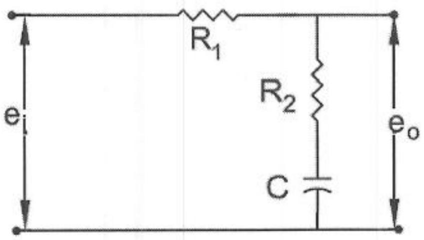
vii) $\tau = 1/\,\omega_m\sqrt{\alpha} = 1/4.6\sqrt{0.21} = 0.474$

viii) $\alpha\tau = 0.21 \times 0.474 = 0.10$

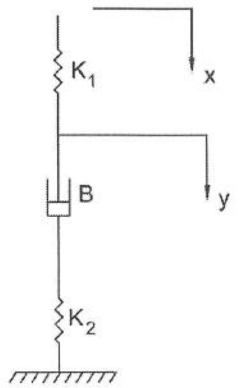
Hence, $G_e(s) = \frac{1+\tau s}{1+\alpha\tau s} = \left(\frac{1+0.47\,S}{1+0.103\,S}\right)$

The compensated plant is $G_c(s)\, G(s) = \left(\frac{1+0.47\,S}{1+0.10\,S}\right)^2 \frac{(10)}{s^2(0.2S+1)}$

Lag Compensator can be realized as shown



Mechanical Equivalent for lag realisation



$$\frac{Eo(s)}{Ei(s)} = \frac{(R2+\frac{1}{sc})}{\mathrm{R1}+\mathrm{R2}+\frac{1}{sc}} \frac{y(s)}{x(s)} = \frac{1+\tau s}{1+\beta\tau s}$$

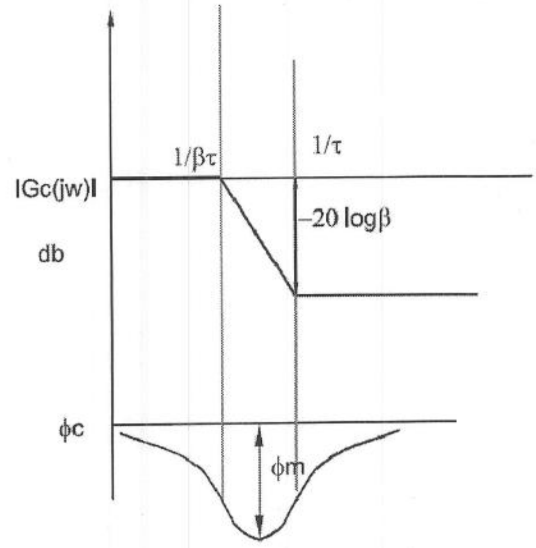
Letting τ = R_2 C ; β = (R_1 + R_2) /R_2> 1 where $\tau = \frac{B}{k_2}$

The transfer function becomes $G_c(s)$ = $\frac{1+\tau s}{1+\beta\tau s}$

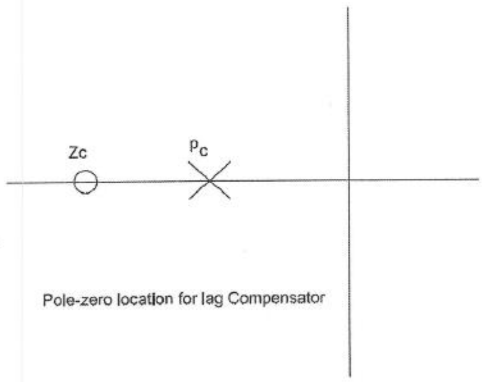
Or, pole zero form as $G_c(s) = \frac{1}{\beta}(\frac{s+\frac{1}{\tau}}{s+\frac{1}{\beta\tau}}) = \frac{1}{\beta}\frac{s+Zc}{s+Pc}$ρ = $(K_1+K_2)/K_1$ >1

$\emptyset_c = tan^{-1}\omega\tau - tan^{-1}\beta\omega\tau$ which is – ve as β>1

Hence the contribution to phase angle of $G_f(c)$ is negative



Bode Plot for lag Compensator



Pole-zero location for lag Compensator

Note: Relations for Φ_{max} (at the geometric mean of = $\frac{1}{\beta\tau}$& = $\frac{1}{\tau}$) etc. are similar to lead compensator.

Note: Addition of lag (or decreasing the phase difference) is undesirable. However, **the characteristic taken advantage is - high frequency gain attenuation (reduction of gain by 20log β), at frequency where Φ due to $G_c(s)$ is very little.**

Lag Compensator: Design Procedure

Lag compensator can be used only when there is a range of Φ more than -180^0 in G(s), since phase lag increases due to it. The Compensator relies on reducing the gain of the compensated plant in the critical region to make it adequately stable.

i) Find the open-loop gain necessary to satisfy the required condition

ii) Find the frequency ω_{c2} when uncompensated system makes

$\varnothing_2 = \varnothing_s + \varepsilon$ where ε is 5^0 to 15^0 to account for phase lag due to network

iii) Measure the gain of UC system at ω_{c2}, and equate it to 20 log β and find β. Choose upper corner frequency $\omega_2 (= 1/\tau)$ of network one octave or decade below ω_{c2}, ($\omega_2 = \omega_{c2}/2$ ie. Octave or $\omega_{c2}/10$ for decade)

iv) Knowing $\beta \& \tau$, ω_2 $(= 1/\tau)$; ω_1 $(= 1/\beta\tau)$; $G_c = \frac{1+\tau s}{1+\beta\tau s}$

Following example will clarify:

Example: Design a lag compensator so that the plant $G_f(s) = \frac{K}{s(s+1)(s+4)}$ meets the specifications: $\xi = 0.4$ and $K_v = 5$;

Solution:

Step1: $K_v = s\,G(s)|_{s \to 0} = K/4 \Rightarrow K = 20$;

Using approximation → $\varnothing_s = 100\,\xi$,

$\varnothing_s = 0.4 \times 100 = 40^0$

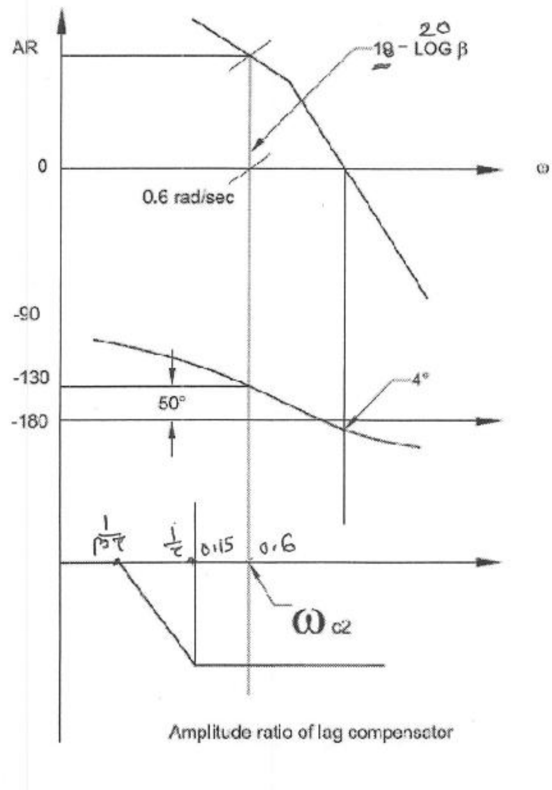
Step2: Bode plot for the system

$G(s) = \frac{20}{s(s+1)(s+4)}$ is as shown.

When amplitude ratio, in log terms $|G(j\omega)|$ is zero,

the Φ value is 4^0 below -180^0,

Φ < -180^0 and the system is unstable.



Amplitude ratio of lag compensator

Step3: To the required $\Phi_s = 40^0$, add $\varepsilon = 5^0$ to compensate for negative Φ_s.

$\Phi = 40+5 = 45^0$

Step 4: Frequency where $\Phi = -135^0$ is 0.6 rad/sec.

Choose ω_2 at 2 octaves less ,

ie., $\omega_2 = 0.6/2^2 = 0.15$ rad/sec $= 1/\tau$

ie., $\tau = 6.67$

Step 5: Gain at this frequency = 18 dB, which has to become 0, when $\Phi = -135^0$

Hence $20 \log \beta = 18$, or $\beta = 7.94$

$\beta\tau = 7.94 \times 6.67 = 53$

Step 6: Hence the compensator to be implemented is

$$G_c(s) = \frac{1+\tau s}{1+\beta\tau s} = \frac{(1+6.67s)}{(1+53\,s)}$$

The compensated plant is

$$G_c(s)\,G(s) = \frac{(1+6.67s)}{(1+53\,s)}\frac{20}{s(s+1)(s+4)}$$

Compensator Design – Root Locus Approach (Time-domain)

Lead Compensator design procedure::

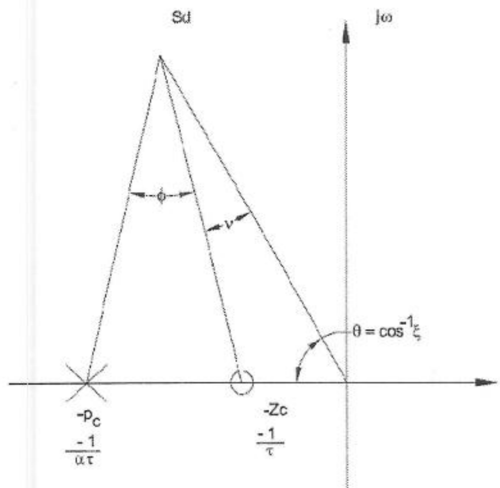
Translate the desired specifications into a dominant pair of complex poles in the Root Locus. (complex plane). If the given system $G_f(s)$is such that

$\measuredangle G_f(s) \neq \pm 180^0$ at $S_d\angle$ then a compensator needs to be designed

such that∟∠

$\measuredangle\ G_f(s)\ G_c(s) \neq \pm 180^0$ at Sd or $\measuredangle G_c(s)_d = \pm 180^0 - G_f(s)_d = \Phi$ ------ (A)

ie. the pole-zero pair of compensator have to contribute Φ.



Lead Compensator
Pole-zero locations

From the above figure, we have from properties of Δ^s,

$z_c/\omega_n = \frac{\sin\gamma}{\sin(\pi-\theta-\gamma)}$ and $p_c/\omega_n = \frac{\sin(\gamma+\Phi)}{\sin(\pi-\theta-\gamma-\Phi)}$

So that $z_c/p_c = \frac{\sin\gamma \sin(\pi-\theta-\gamma-\Phi)}{\sin(\pi-\theta-\gamma)\sin(\gamma+\Phi)}$

Φ and $d\propto/d\gamma = z_c/p_c = \sin\gamma$

(to reduce the effect of gain due to compensator, it is desirable to have large $\propto$).

which leads to $\gamma = ½(\pi - \theta - \Phi)$ --------(B)

Thus after locating Sd in the complex plane, knowing $\theta = \cos^{-1}\xi$, and Ø from (A), locate z_c from (B), as well as p_c

Knowing z_c and p_c, the compensator is $(s+z_c)/(s+p_c)$

Example – Lead Compensator design - Time domain

Design a compensator is give ξ= 0.5 and ω_n = 2 for the system $G_f(s) = \frac{K}{s(s+1)(s+4)}$

Solution: Given the damping ratio ξ= 0.5(=$Cos^{-1}\theta$) and ω_n of the dominant complex pole pair, the point S_d to pass through the root locus of the compensated plant is

$Sd= - \xi\omega_n \pm j\ \omega_n\sqrt{(1-\xi^2)}$= -1 $\pm$ j 1.73

θ= Cos^{-1} 0.5 = 60^0

Φ = -180 – $G_f(S_d)$

= -180^0 –(-120-90-60)= 60^0

ν= ½ $(\pi-\theta-\Phi)$= $(\pi-60-60)/2$= 30^0

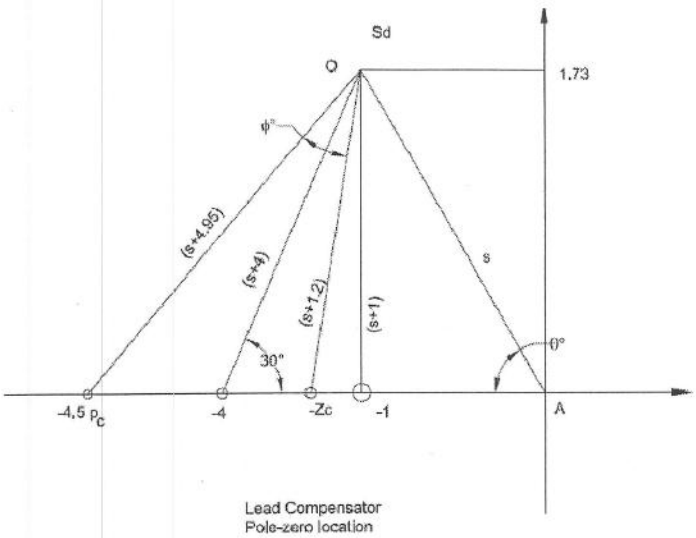
However, locating oz so that ν =30^0 makes Z_c coincide with the pole of (-1,0) of the given system.

Hence choose Z_c = -1.2 close to -1

Now draw OP_c so that angle Z_c O P_c = Φ = 60^0

P_c cuts the real axis at -4.95 so that

$G_c=(\frac{S+Zc}{s+Pc}) = \frac{(S+1.2)}{(s+4.95)}$



Lead Compensator
Pole-zero location

Also the gain at Sd(= 1+ j 1.73) for the compensated system is

K = |s|. |s+1| . |s+4| . |s+ 4.95|/|s+1.2| ==30, in which |s|, |s+1| etc are the lengths of vectors which can be measured and substituted.

Hence compensated system = $G_c(s)G(s)$ = 30(s +1.2) / s(s+1) (s+4) (s+4.95)

Lag Compensator – Time Domain (or Root Locus approach)

Method: Lag compensator is designed when the steady – state response needs improvement, while transient performance is satisfactory.

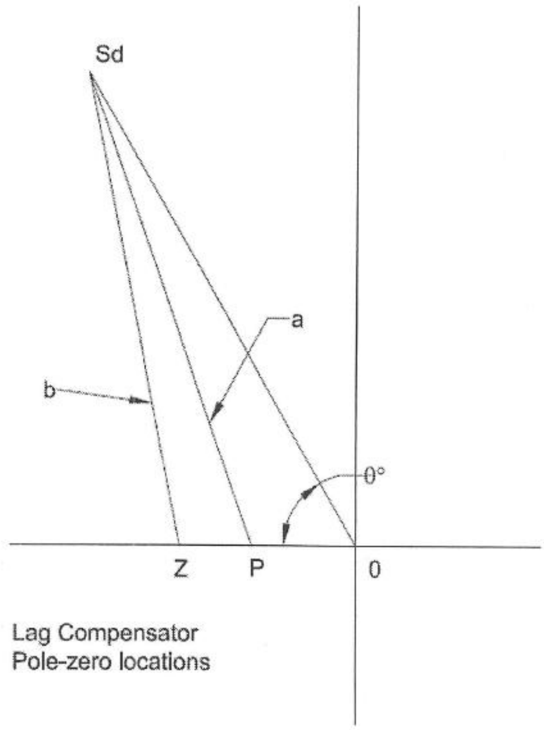
As the transient performance is satisfactory, the technique lies in ensuring that the root locus passes through the same Sd as for the uncompensated system while the error constant is improved.

Therefore pole P is chosen close to origin and zero to its left so that $\beta = Z_c / P_c > 1$

where $\Phi = \measuredangle PSZ$ is 2 to 5^0

and $\measuredangle OSZ < 10^0$,

We can establish value of β using the following reasoning:



Lag Compensator
Pole-zero locations

Gain of UC system at Sd

$$= K^{uc}(Sd) = \frac{|Sd|r\,.|\,Sd + P1|.|\,Sd + P2\,|.....}{|Sd + Z1\,|.|\,Sd + Z2\,|\,...}$$

$$\text{Also, } K^{c}(Sd) = \frac{|Sd|r\,.|\,Sd + P1|.|\,Sd + P2\,|.....}{|Sd + Z1\,|.|\,Sd + Z2\,|\,...} \times \frac{a}{b}$$

Since $a \cong b$, we have $K^{uc}(S_d) = K^{c}(S_d)$

$$\text{Now, Error Constant } K_e^{c} = K^{c}(S_d)\,\frac{m\,Zi}{n\,Pi} \times \frac{Zc}{Pc}$$

$$= K^{uc}(S_d)\,\frac{m\,Zi}{n\,Pi} \times \frac{Zc}{Pc}$$

$$\cong K_e^{uc}\ .\ \frac{Zc}{Pc}$$

Hence $\beta = \frac{Zc}{Pc} = K_e^{c} / K_e^{uc}$

Thus β can be chosen as the ratio of the expected Error Constant to the Error Constant of the given system

Example: Design a lag compensator for the system

$G_f(s) = \frac{K}{s(s+1)(s+4)}$ to meet the following specs:

Damping ratio = ζ = 0.5; settling time t_s =10 secs & $K_v \geq 5\ sec^{-1}$

Solution:

Step 1: From ts = 10 = $4/\zeta\omega n$ => ωn = 4/ 10 x 0.5 = 0.8

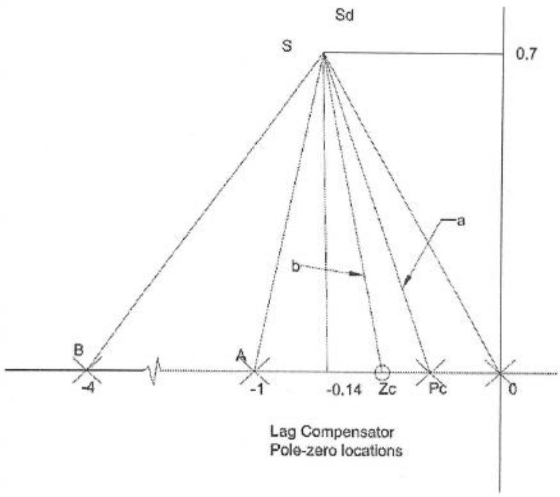
Thus Sd = - $\zeta\omega_n$ + j $\sqrt{1-\zeta^2}\omega_n$ = - 0.4 $\pm$ j 0.7
S_d lies on the uncompensated system as well.

Step2: At S_d, gain of the UC system is

K^{uc} = OS x SA x SB= 0.8 x 0.9 x 3.7 = 2.66
Velocity Error Constant of UC system

K_v^{uc} = | s G (s) | = 2.66/(1 x 4) = 0.666

s→0



Lag Compensator
Pole-zero locations

Step 3: Thus β = K_v desired / K_v existing

= 5/0.666 = 7.5

Say β = 10 to counter the effect of negative phase difference due to lag compensator.

Step 4: Now to locate the pole and zero,

Draw a line from S to Z_c so that ∡OSZ_c= 6^0 . (small value)

Determine OZ_e = 0.1 (measure)

Since β = Z_c/P_c, P_c = Z/10 = 0.1/10 = 0.01

Also as the gain compensated system may be treated as the same as of UC system – in which case

$G_f G_c = \frac{2.66\,(s+0.1)}{s(s+1)(s+4)(s+0.01)}$, which is the Compensated plant transfer function

Or get K = $\frac{|OS|.|SA|.|SB|.|SPc|}{|SZc\,|}$ = 2.2 instead of 2.66,

by measuring the line lengths from the figure.

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